Cauchy Theorem—Multiply connected case; Cauchy integral formula

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References: Lecture Notes on Cauchy formula and its consequences; Sec. 4.4, 4.5 of the textbook.

Cauchy Theorem for multiply connected domain:
Assume $D$ has outer boundary $\Gamma$ and inner boundaries $C_1, \ldots, C_n$, with counterclockwise direction. Suppose $f$ is analytic in $D$. Then:

$$\int_{\Gamma} f(z) = \int_{C_1} f(z) \, dz + \ldots + \int_{C_n} f(z) \, dz.$$ 

Ex1. $\int_{\Gamma} \frac{dz}{z^2 - \frac{1}{4}}$, $\Gamma$ is $|z| = 2$ with counter-clockwise orientation.

Let $C_1$ be $\left| z + \frac{1}{2} \right| = \frac{1}{4}$, $C_2$ be $\left| z - \frac{1}{2} \right| = \frac{1}{4}$, with counter-clockwise orientation.

Then

$$\int_{\Gamma} \frac{1}{z^2 - \frac{1}{4}} \, dz = \int_{C_1} \frac{1}{z^2 - \frac{1}{4}} \, dz + \int_{C_2} \frac{1}{z^2 - \frac{1}{4}} \, dz.$$ 

For $C_1$, we have

$$\int_{C_1} \frac{1}{z^2 - \frac{1}{4}} \, dz = \int_{C_1} \left( \frac{1}{z - \frac{1}{2}} - \frac{1}{z + \frac{1}{2}} \right) \, dz = \int_{C_1} \frac{1}{z - \frac{1}{2}} \, dz - \int_{C_1} \frac{1}{z + \frac{1}{2}} \, dz = 0 - 2\pi i = -2\pi i.$$ 

Similarly for $C_2$, we have

$$\int_{C_2} \frac{1}{z^2 - \frac{1}{4}} \, dz = \int_{C_2} \left( \frac{1}{z - \frac{1}{2}} - \frac{1}{z + \frac{1}{2}} \right) \, dz = \int_{C_2} \frac{1}{z - \frac{1}{2}} \, dz - \int_{C_2} \frac{1}{z + \frac{1}{2}} \, dz = 2\pi i.$$ 

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Hence

\[ \int_{\Gamma} \frac{1}{z^2 - \frac{1}{4}} \, dz = 2\pi i - 2\pi i = 0. \]

Cauchy integral formula:

Let \( \Gamma \) be a simple closed curve (with counter-clockwise direction) and \( z_0 \) is inside \( \Gamma \). Suppose \( f \) is analytic inside \( \Gamma \). Then

\[ \int_{\Gamma} \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0). \]

Sketch of the proof: Let \( C_r \) be the circle \( |z - z_0| = r \). By Cauchy Theorem, we have

\[ \int_{\Gamma} \frac{f(z)}{z - z_0} \, dz = \int_{C_r} \frac{f(z)}{z - z_0} \, dz. \]

As \( r \to 0 \), \( f(r) = f(z_0) + f'(z_0) (z - z_0) + O\left(|z - z_0|^2\right) \). (To be more precise, the last term, denoted by \( I \), will satisfy

\[ \frac{I}{|z - z_0|} \to 0, \text{ as } |z - z_0| \to 0. \]

Hence, as \( r \to 0 \),

\[ \int_{C_r} \frac{f(z)}{z - z_0} \, dz \to f(z_0) \int_{C_r} \frac{1}{z - z_0} \, dz = 2\pi i f(z_0). \]

Remark: To prove that \( \int_{C_r} \frac{f(z)}{z - z_0} \, dz \) tends to \( 2\pi i \) as \( r \to 0 \), we actually only need to assume that \( f \) is continuous at \( z_0 \).

Ex2. \( \int_{\Gamma} \frac{e^z}{z} \, dz \), where \( \Gamma \) is \( |z| = 2 \) with counter-clockwise direction.

Applying the Cauchy integral formula (Taking \( f(z) = e^z, z_0 = 1 \), we get

\[ \int_{\Gamma} \frac{e^z}{z - 1} \, dz = 2\pi i e^1 \]

\[ = 2\pi i. \]