1. Let $X$ be a r.v. taking values in $\{1, \ldots, 6\}$ with p.m.f. of the form

$$P(X = k) = ck.$$ (a) Find $c$.
(b) Find the c.d.f. of $X$.

**Solution.**
(a) Since $\sum_{i=1}^{6} p(i) = 1$, we find $c = \frac{1}{21}$.
(b) One way to write this:

$$F(t) = \begin{cases} 
0 & t < 1, \\
\frac{1}{21} & 1 \leq t < 2, \\
\frac{3}{21} & 2 \leq t < 3, \\
\frac{6}{21} & 3 \leq t < 4, \\
\frac{10}{21} & 4 \leq t < 5, \\
\frac{15}{21} & 5 \leq t < 6, \\
1 & 6 \leq t.
\end{cases}$$

For $0 \leq t < 7$ this can also be written as $F(t) = \lceil t \rceil (\lceil t \rceil - 1)/21$, where $\lceil t \rceil$ is the integer part of $t$.

2. Suppose that the continuous RV $X$ has c.d.f. given by

$$F(x) = \begin{cases} 
0 & x < \frac{1}{\sqrt{2}} \\
5 - 12\sqrt{2}x + 18x^2 - 4\sqrt{2}x^3 & \frac{1}{\sqrt{2}} \leq x < \sqrt{2} \\
1 & \sqrt{2} \leq x
\end{cases}$$

(a) Find the smallest interval $[a, b]$ such that $P(a \leq X \leq b) = 1$.
(b) Find $P(0 < X < \frac{1}{2})$.
(c) Find $P(X = 1)$.
(d) Find $P(1 \leq X \leq \frac{3}{2})$.
(e) Find the p.d.f. of $X$.

**Solution.**
(a) $a = 1/\sqrt{2}$ and $b = \sqrt{2}$.
(b) This is $F(1/2) - F(0) = 0 - 0$.
(c) $P(X = a) = 0$ for every $a$.
(d) This is $F(3/2) - F(1) = 1 - F(1) = -22 + 16\sqrt{2}$.
(e) $f(x) = F'(x) = -12\sqrt{2} + 36x - 12\sqrt{2}x^2$ on $[1/\sqrt{2}, \sqrt{2}]$ and 0 outside this interval.

3. Let $X$ be a random variable with p.d.f.

$$f(x) = \begin{cases} 
2x^{-2} & x > 2 \\
0 & \text{otherwise}
\end{cases}$$

(a) Compute the c.d.f. of $X$.
(b) Find $P(X > 3)$.
(c) Find $P(X > 3 | X < 5)$. 

Solution.

(a) \[ F(b) = \int_{-\infty}^{-b} f(x) \, dx = \begin{cases} 0 & b < 2 \\ \int_{2}^{b} 2x^{-2} & b \geq 2 \end{cases} = \begin{cases} 0 & b < 2 \\ 1 - 2b^{-1} & b \geq 2 \end{cases} \]

(b) \[ P(X > 3) = 1 - F(3) = \frac{2}{3} \]

(c) \[ P(X > 3 \mid X < 5) = P(\{X > 3\} \cap \{X < 5\}) = \frac{P(X \in (3, 5))}{P(X < 5)} = \frac{F(5) - F(3)}{F(5)} = \frac{4}{9} \]

4. Define the function
\[ f(x) = \begin{cases} 9x^2 - 4x^3 + b & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]
Show that there is no value of \( b \) for which this is the p.d.f. of some continuous RV.

Solution. We must have \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \). This is \( \int_{0}^{1} 9x^2 - 4x^3 + b \, dx = 2 + b \), so \( b = -1 \). However, this means that \( f(x) < 0 \) for some values of \( x \) (any \( x \in [0, 1/3] \) for example.).

5. A stick of length \( \ell \) is broken into two pieces at a position \( X \sim \text{Unif}[0, \ell] \). Let \( Y \) denote the length of the smaller piece.
   (a) Calculate the c.d.f. of \( Y \), that is, calculate \( P(Y \leq b) \).
   (b) Calculate the p.d.f. of \( Y \). Can you identify what kind of random variable \( Y \) is?

Solution.

(a) The smaller segment can be anything from 0 to \( \ell/2 \). In order to get \( Y \leq b \) the uniform point \( X \) must be within \( \ell \) of either end of the stick, so \( F(b) = P(Y \leq b) = 2b/\ell \) for \( 0 \leq b \leq \ell \). It is 0 or 1 elsewhere.

(b) The pdf is \( F'(b) = \begin{cases} 2/\ell & 0 \leq b \leq \ell/2 \\ 0 & \text{otherwise} \end{cases} \). This means \( Y \) is uniform on \( [0, \ell/2] \).

6. Let \( X \) be an Exp(4) random variable. Find a number \( a \) such that \( \{X \in [0, 1]\} \) is independent of \( \{X \in [a, 2]\} \)

Solution. If \( a < 0 \) then all probabilities are the same as in the case \( a = 0 \), so we may assume that \( a \geq 0 \). If \( a > 1 \) then the events are disjoint (and hence not independent), so we may further assume that \( 0 \leq a \leq 1 \). We have
\[ P(X \in [0, 1]) = F_X(1) - F_X(0) = 1 - e^{-4}, \]
and
\[ P(X \in [a, 2]) = F_X(2) - F_X(a) = e^{-4a} - e^{-8}. \]
The probability of intersection is
\[ P(X \in [0, 1], X \in [a, 2]) = P(X \in [a, 1]) = F_X(1) - F_X(a) = e^{-4a} - e^{-4}. \]
The definition of independence gives the equation
\[ e^{-4a} - e^{-4} = (1 - e^{-4})(e^{-4a} - e^{-8}), \]
so
\[ e^{-4a} = 1 - e^{-4}(1 - e^{-4}), \]
that is,
\[ a = -\frac{1}{4} \ln(1 - e^{-4}(1 - e^{-4})) \approx 0.0045. \]

7. Let \( X \sim \mathcal{N}(0, 1) \). Find \( c \) such that \( P(X > c) \approx \frac{1}{3} \).
You can find the \( \Phi \)-table here: https://en.wikipedia.org/wiki/Standard_normal_table#Cumulative

**Solution.** We have to solve
\[ P(X > c) = 1 - \Phi(c) = \frac{1}{3}, \]
or, in other words, \( \Phi(c) = \frac{2}{3} \). Looking at the table, we see that \( \Phi(0.43) = 0.66640 \), so \( c \approx 0.43 \).