1. Consider the following game: An urn contains 20 white balls and 10 black balls. If you draw a white ball, you get $1, but if you draw a black ball, you lose $2.

(a) You draw 6 balls out of the urn (without replacement). What is the probability that you break even exactly?

(b) You draw 6 balls out of the urn (without replacement). What is the probability that you will win money? (break-even does not count)

(c) How many balls should you draw in order to maximize the probability of winning? Hint: Use a computer.

Solution. (a) The total number of drawings is \( \binom{30}{6} \). You will break even if you draw exactly 4 white balls (and therefore 2 black balls). Thus,

\[
P(\text{break even}) = \frac{\binom{20}{4} \binom{10}{2}}{\binom{30}{6}} = \frac{969}{2639} \approx 0.367.
\]

(b) You will win money if you draw either 5 or 6 white balls. Therefore,

\[
P(\text{win money}) = \frac{\binom{20}{5} \binom{10}{1}}{\binom{30}{6}} + \frac{\binom{20}{6} \binom{10}{0}}{\binom{30}{6}} = \frac{2584}{7917} \approx 0.33.
\]

(c) If \( n \) is the number of balls you decide to draw, the condition on the number of white balls that need to be present in order for you to make money is \( k - 2(n - k) > 0 \), or \( k > \frac{2}{3}n \). Therefore,

\[
P(\text{win money with } n \text{ draws}) = \sum_{k > \frac{2}{3}n}^{n} \frac{\binom{20}{k} \binom{10}{n-k}}{\binom{30}{6}}.
\]

Evaluating these values numerically for all possible \( n = 0, \ldots, 30 \), we see that the largest value is achieved at \( n = 1 \), i.e. only choosing a single ball, where the probability of winning is \( \frac{2}{3} \).

2. A chess rook can move any distance in a row or a column of the chess board. Eight rooks are placed randomly on an 8 \( \times \) 8 chess board. What is the probability that none of the rooks can capture any of the other rooks? (In non-chess terms: Randomly pick 8 squares from the board. What is the probability that no two squares share a row or a column?)

Hint: How many choices do you have to place rooks in the first row? After you have made your choice, how many choices do you have for the second? Continue this reasoning.

Solution. The total number of choosing 8 positions for the rooks on a board with 64 fields is \( \binom{64}{8} \). The number of favorable outcomes is \( 8! \): there are 8 possibilities two choose a square from the first row, 7 ways to choose one from the second row, and so on. Thus our probability in question is

\[
P = \frac{8!}{\binom{64}{8}} = \frac{(8!)^2}{64 \cdots 57}.
\]

3. An experiment is repeated, and the first success occurs on the 5th attempt. What is the success probability for which this is most likely to happen? (This is called the maximum likelyhood estimator.)
Solution. Let $p$ denote the success probability. Then

$$P(\text{first success on 6th attempt}) = p(1-p)^5.$$  

Denote $f(p) = (1-p)^5p$. Then its derivative is

$$f'(p) = (1-p)^5 - 5p(1-p)^4.$$  

Solving for $f'(p) = 0$, we find that $1 - p = 5p$ so that $p = \frac{1}{6}$. A simple computation shows that $f''(\frac{1}{6}) < 0$ so that $p = \frac{1}{6}$ is a maximum point for $f$. Thus, $p = \frac{1}{6}$ is the success probability for which the above is most likely to happen.

4. An experiment is repeated 10 times, and succeeds 4 out of the 10. What is the success probability for which this is most likely to happen?

Solution. Let $p$ denote the success probability. Then

$$P(4 \text{ of 10 successes}) = \binom{10}{4} p^4(1-p)^6 = 210p^4(1-p)^6.$$  

Denote $c = 210$ and $f(p) = cp^4(1-p)^6$. Then its derivative is

$$f'(p) = 4cp^3(1-p)^6 - 6cp^4(1-p)^5.$$  

Solving for $f'(p) = 0$, we find that $4(1-p) = 6p$ so that $p = \frac{2}{5}$. A simple computation shows that $f''(\frac{2}{5}) < 0$ so that $p = \frac{2}{5}$ is a maximum point for $f$. Thus, $p = \frac{2}{5}$ is the success probability for which the above is most likely to happen.

5. An exam question asked for the probability that a Bin(7, $p$) random variable is equal to 3, with a certain given $p$. A confused student found instead the probability that a Geom($p$) random variable is equal to 3. Miraculously, the student got the correct result. What can be the value of $p$?

Solution. Let $X$ be a Bin(7, $p$) random variable and let $Y$ be a Geom($p$) random variable. Then

$$P(X = 3) = \binom{7}{3} p^3(1-p)^4 = 35p^3(1-p)^4 \quad \text{and} \quad P(Y = 3) = p(1-p)^2.$$  

Let us find the solutions to $35p^3(1-p)^4 = p(1-p)^2$. Setting aside the trivial solutions $p = 0$ and $p = 1$, we may equivalently find the solutions to $35p^2(1-p)^2 = 1$. Thus, $p(1-p) = \frac{1}{\sqrt{35}}$, or equivalently, $p^2 - p + \frac{1}{\sqrt{35}} = 0$. The solutions of this quadratic equation are

$$p = \frac{1 \pm \sqrt{1 - 4/\sqrt{35}}}{2},$$

both of which are between 0 and 1, and are therefore possible values of $p$.

6. Pick a uniformly chosen random point $(X, Y)$ inside a unit square $[0, 1] \times [0, 1]$, and let $M = \min(X, Y)$. Find the probability that $M < 0.4$. 

Solution. The sample space is $S = [0, 1]^2$ and the event in question is $E = \{M < 0.4\}$. Note that its complement is $E^c = \{(x, y) : 0.4 \leq x, y \leq 1\}$. Thus, $E^c$ describes a square whose area is $0.6^2 = 0.36$. Since the area of $[0, 1]^2$ is 1, we obtain that

$$P(E) = 1 - P(E^c) = 1 - \frac{0.36}{1} = 0.64.$$  

7. (a) Prove that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ for any integer $n \geq 1$.

(b) If $n$ fair coins are tossed, prove that

$$P(\text{the number of heads is even}) = \frac{1}{2}.$$  

Solution. (a) Recall that the Binomial theorem states that $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x + y)^n$. Thus,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = ((-1) + 1)^n = 0.$$  

(b) Observe that

$$P(\text{number of heads is even}) = \sum_{k=0, \text{even}}^{n} \binom{n}{k} 2^{-n}$$  

and

$$P(\text{number of heads is odd}) = \sum_{k=0, \text{odd}}^{n} \binom{n}{k} 2^{-n}.$$  

Thus, by part (a),

$$0 = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k = \sum_{k=0, \text{even}}^{n} \binom{n}{k} 2^{-n} - \sum_{k=0, \text{odd}}^{n} \binom{n}{k} 2^{-n} = P(\text{number of heads is even}) - P(\text{number of heads is odd}).$$  

This shows that

$$P(\text{number of heads is even}) = P(\text{number of heads is odd}).$$  

On the other hand,

$$P(\text{number of heads is even}) + P(\text{number of heads is odd}) = 1.$$  

Therefore,

$$2P(\text{number of heads is even}) = 1,$$

so that

$$P(\text{number of heads is even}) = \frac{1}{2}.$$  

Extra practice problems (do not hand in):
Chapter 1: 28, 33, 35, 45, 46.