Last week: defined a probability (three axioms)

Today:
(1) Simple consequences of axioms
(2) Acquire basic counting/combinatoric skills.

Easy consequences of axioms:

(i) For a finite number of pairwise disjoint events $E_1, E_2, \ldots, E_n$, we have

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n)$$

Proof: Set $E_{n+1} = E_{n+2} = \cdots = \emptyset$. 

By axiom (III):

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

\[ P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) \]
(ii) Suppose you know \( P(E) \), then \( P(E^c) = 1 - P(E) \).

Proof:

\[
1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)
\]

by axiom (II)

axiom (III)

(iii) If \( E \subseteq F \), then \( P(E) \leq P(F) \).

Proof:

\[
P(F) = P(E \cup (F \setminus E)) = P(E) + P(F \setminus E) \geq 0
\]

by axiom (I)

axiom (III)

\[
F \setminus E = \{ x \in S : x \in F, x \in E \}
\]

\[
F \setminus E = F \cap E^c
\]
(iv) For any events $E$ and $F$:

$$P(EUF) = P(E) + P(F) - P(ENF)$$

**Proof:**

$EUF = E \cup (F \cap E)$

$F = (ENF) \cup (F \cap E)$

Axiom (iii) \[\rightarrow\]

$$P(EUF) = P(E) + P(F \cap E)$$

$$P(F) = P(ENF) + P(F \cap E)$$

Now subtract equations.

**Essential Combinatorics**

**Example:** Every day, Monday - Friday, given a choice of 1 of 3 fruits.

How many "weekly menus" can you create?

$$\text{Sale} \quad M \quad T \quad W \quad T \quad F \quad \boxed{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = 3^5 = 243$$
Rule: The number of ways to choose an ordered sequence of $k$ elements from a set of size $n$ with replacement is $n^k$.

Example: How many postal codes exist in Canada?

Sol: The postal code consists of 3 letter and 3 numbers.

$26 \cdot 10 \cdot 26 \cdot 10 \cdot 26 \cdot 10$

$= \frac{26^3 \cdot 10^3}{17 \text{ million}}$

Rule: The number of ways to choose an order sequence of $k$ elements with replacement, where the $i$-th element is chosen from a set of size $n_i$, is $n_1 \cdot n_2 \cdot n_3 \cdots n_k$. 
Example: In how many ways can we elect a President, Secretary, Treasurer from a group of 30 people? (no one can hold two positions)

Sol:

\[
PST
30 \cdot 29 \cdot 28 = 30 \cdot 29 \cdot 28 = 124,360
\]

Alternatively:

\[
STP
30 \cdot 29 \cdot 28 = \ldots = \text{same}
\]

Rule:

The number of ways to choose an ordered sequence of k elements from a set of size n without replacement is

\[
\binom{n}{k} = \frac{n!}{(n-k)!}
\]

Note: \(n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}\), \(m! = m(m-1)(m-2)\ldots2\cdot1\).
Example:  \# of ways to elect a committee of 3 from 10 people?

Sol: \[
\frac{10!}{7!} = 10 \cdot 9 \cdot 8 = 720 \quad \text{\# of ways to elect President, Secretary, Treasurer.}
\]

overcounting: each set of 3 committee members \{a, b, c\} was counted 6 times

\( (a, b, c) \quad (b, a, c) \quad (c, a, b) \)
\( (a, c, b) \quad (b, c, a) \quad (c, b, a) \)

So solution is \[
\frac{720}{6} = 120
\]
Rule: \# of ways to choose an unordered set of size \( k \) from a set of size \( n \) without replacement is \( \binom{n}{k} = \frac{n!}{(n-k)! k!} \) "n choose k"

Example: How many lottery tickets must you fill out to be certain to win? (7 numbered balls are chosen from a bin of 49)

\[
\text{Sol: } \binom{49}{7} \approx 85.9 \text{ million}
\]
Last time: Consequences of axioms
Basic combinatorics

Today: More combinatorics

Exercise: Poker - why does "4 of a kind" beat "full house"?

Sol: deck of cards has 52 cards:
numbers: 1, 2, ..., 13
suits: heart, spade, club, diamond

Each player receives 5 cards
(order does not matter)
aaaabbbb baaaaa
# of hands with "4 of a kind" =
\[ \binom{13}{1} \cdot \binom{48}{12} = 13 \cdot 12 = 156 \]

# of hands with "full house" =
\[ \binom{13}{1} \cdot \binom{48}{12} = 13 \cdot 12 \cdot 4 \cdot 6 = 3744 \]

"full house" is 6 times more likely than "4 of a kind".
Exercise: In many ways can you choose 4 balls from a bin of red and green and blue balls (unlimited supply of each)?

Sol: all same color $\Rightarrow 3$
all three colors $\Rightarrow 3$

exactly two colors

$\Rightarrow 3 + 3 + 3 + 6 = 15$

aaab $\Rightarrow \binom{3}{1} \cdot \binom{3}{1} = 6$
choose a

choose b
Rule: \# of ways to choose an unordered set of \( k \) elements from a set of size \( n \) with replacement is \( \binom{n+k-1}{k} \).

In the exercise: \( k=4 \), \( n=3 \) \( \Rightarrow \binom{3+4-1}{4} = \binom{6}{4} \)
\[ \frac{6!}{4!(6-4)!} = \frac{6 \cdot 5}{2} = 15 \]

Summary:

\# of ways to choose \( k \) elements from a set of size \( n \)

<table>
<thead>
<tr>
<th>Ordered</th>
<th>with replacement</th>
<th>( n^k )</th>
<th>without replacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unordered</td>
<td>( \binom{n+k-1}{k} )</td>
<td>( \binom{n}{k} )</td>
<td>( \binom{n}{k} )</td>
</tr>
</tbody>
</table>
Binomial theorem: \( \forall x, y \in \mathbb{R}, \ n \ \text{positive integer} \)

\[
(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\]

Example:

\[
(x+y)^3 = 1 \cdot x^3 + 3 x^2 y + 3 x y^2 + y^3.
\]

Explanation for theorem:

\[
(x+y)^n = (x+y) \cdot (x+y) \cdot (x+y) \cdots (x+y)
\]

How many terms of the form \( x^k y^{n-k} \) are there?

Choose \( k \) terms for \( "x" \):

\[
\binom{n}{k}.
\]
application: How many subsets of \{1, 2, \ldots, n\} are there?
(including the empty set and the entire set)

Sol:

\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n-1} + \binom{n}{n} = \\
= \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} \\
= (1+1)^n = \boxed{2^n}
\]

Another solution: for each element \(i \in \{1, 2, \ldots, n\}\), we have two options: include or not include.

\Rightarrow 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n
Last time: Basic combinatorics
Binomial theorem

Today: Multinomial theorem
Urн problems

Exercise: Every day, Monday–Sunday, given a choice of 1 of 3 fruits (apple, orange, banana). A healthy menu consists of 3 apples, 2 oranges, 2 bananas. How many “healthy weekly menus” can we create?

Sol: 

Monday Tuesday Wednesday Thursday Friday Saturday Sunday

apple banana orange

Choose days for bananas: \( \binom{7}{2} \)
Choose days for apples: \( \binom{5}{2} \)
Choose days for oranges: \( \binom{3}{2} \)

\[ \binom{7}{2} \cdot \binom{5}{2} \cdot \binom{3}{2} = 21 \cdot 10 \cdot 1 = 210 \]
More generally: given $n$ days, $n_1$ fruits "A"
$n_2$ fruits "B"
$\vdots$
$n_m$ fruits "Z"

$\# \text{ healthy menus} = \binom{n}{n_1} \cdot \frac{n!}{n_1! (n-n_1)!} \cdot \binom{n-n_1}{n_2} \cdot \frac{(n-n_1)!}{n_2! (n-n_1-n_2)!} \cdot \binom{n-n_1-n_2}{n_3} \cdot \frac{(n-n_1-n_2)!}{n_3! (n-n_1-n_2-n_3)!} \cdot \ldots \cdot \binom{n-n_1-n_2-\ldots-n_{m-1}}{n_m} \cdot \frac{(n-n_1-n_2-\ldots-n_{m-1})!}{n_m! (n-n_1-n_2-\ldots-n_m)!}$

$$= \frac{n!}{n_1! (n-n_1)! n_2! (n-n_1-n_2)! \ldots n_m! (n-n_1-n_2-\ldots-n_m)!}$$

$0! = 1$

**Rule:** # ways to partition a set of size $n$
into groups of sizes $n_1, n_2, \ldots, n_m$ ($n = n_1 + \ldots + n_m$)
is \( \binom{n}{n_1, n_2, \ldots, n_m} = \frac{n!}{n_1! n_2! \ldots n_m!} \)  \( \rightarrow \text{multinomial coefficient} \)
Note: \( \binom{n}{k} = \binom{n}{k, n-k} = \frac{n!}{k! (n-k)!} \)

Example: Deal a deck of 52 cards to 4 people (13 cards each). How many different “deals” are there?

Solution: \( \binom{52}{13, 13, 13, 13} = \frac{52!}{(13!)^4} \approx 5000 \ldots 000 \frac{292005}{282005} \)

Recall the binomial theorem:

\[(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \]

Rewriting in a different form:
Multinomial theorem: \( \forall x_1, x_2, \ldots, x_m \in \mathbb{R}, n \in \mathbb{N} \)

\[
(x_1 + x_2 + \ldots + x_m)^n = \sum_{k_1, k_2, \ldots, k_m = 0}^{n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}
\]

Explanation:

\((k_1, k_2, \ldots, k_m)\) is the number of ways to obtain the term \(x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}\) when opening parentheses

\((x_1 + \ldots + x_m)^n = (x_1 + \ldots + x_m) \cdot (x_1 + \ldots + x_m) \cdots (x_1 + \ldots + x_m)\)
Urn problems (application of Binomial)

Example: an urn contains 8 black balls and 7 white balls. Choose and remove 5 random balls.

Question: What is the probability that exactly 3 of them are black?
(exactly 2 are white)

Sol: unordered without replacement

Sample space $S = \{ D \in U : 101 = 5 \}$

$U = \{ B_1, B_2, \ldots, B_8, W_1, \ldots, W_7 \}$
\[ N = 15, \quad 15! = (\frac{15!}{5!}) = 3003 \]

Assumption: each of the 3003 possibilities is equally likely (uniform probability).

The event in question: \( E = \) "exactly 3 balls are black"

\[ \begin{align*}
1E1 &= \binom{8}{3} \cdot \binom{7}{2} \\
&= \frac{8!}{3!5!} \cdot \frac{7!}{2!5!} \\
&= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} \cdot \frac{7 \cdot 6}{2} \\
&= 8 \cdot 7 \cdot 7 \cdot 3 = 1176
\end{align*} \]

\[ P(E) = \frac{1E1}{151} = \sqrt{\frac{1176}{3003}} \approx 0.39 = 39\% \]