Joint distributions - the continuous case:

**Definition:** A func. $f: \mathbb{R}^2 \rightarrow (0, \infty)$ is the joint p.d.f. of two cont. r.v. $X_1$ and $X_2$ if

$$P(X_1 \in A_1, X_2 \in A_2) = \iint_{A_1 \times A_2} f(x_1, x_2) \, dx_1 \, dx_2$$

or equivalently, if

$$P((X_1, X_2) \in A) = \iint_{A} f(x_1, x_2) \, dx_1 \, dx_2$$

\( \forall A \subset \mathbb{R}^2 \).
Normalization: \( \int_{\mathbb{R}^2} f(x_1, x_2) \, dx_1 \, dx_2 = 1 \)

**Example:** Suppose that \( X_1 \) and \( X_2 \) have joint p.d.f.

\[
f(x_1, x_2) = \begin{cases} 
x_1 x_2 & \text{if } 0 \leq x_1 \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

(4) Compute \( P(X_2 > \frac{1}{2} \mid X_1 < \frac{1}{2}) \)

\[
P(X_2 > \frac{1}{2} \mid X_1 < \frac{1}{2}) = \frac{P(X_1 < \frac{1}{2}, X_2 > \frac{1}{2})}{P(X_1 < \frac{1}{2})}
\]

\[
= \frac{\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{2} x_1 x_2 \, dx_1 \, dx_2}{\int_{0}^{\frac{1}{2}} \int_{0}^{2} x_1 x_2 \, dx_1 \, dx_2}
\]
\[
\frac{\left( \int_0^x x_1 \, dx_1 \right) \cdot \left( \int_0^{x_2} x_2 \, dx_2 \right)}{\left( \int_0^1 x_1 \, dx_1 \right) \cdot \left( \int_0^{x_2} x_2 \, dx_2 \right)} = \frac{x_2^{1/2}}{x_2^{1/2}} = \frac{y - \frac{1}{2}}{\frac{1}{4}} = \frac{15}{16}.
\]

(x) Compute \( \mathbb{P}(X_1 > X_2) \).

\[
\mathbb{P}(X_1 > X_2) = \iiint_{\{x_1 > x_2\}} f(x_1, x_2) \, dx_1 \, dx_2
\]

\[
= \frac{1}{2} \int_0^1 \left( \int_0^{x_1} x_1 x_2 \, dx_2 \right) \, dx_1
\]

\[
= \frac{1}{2} \int_0^1 x_1 \left( \frac{x_2^2}{2} \right)_{x_2=0}^{x_2=x_1} \, dx_1 = \frac{1}{2} \int_0^1 x_1^3 \, dx_1 = \frac{1}{8}.
\]
Remarks: As with p.d.f of a single cont. r.v.: 

\((*)\) \(f(x_1, x_2)\) is not a probability, only 2D integrals of it are. 
\(f(x_1, x_2)\) measures how likely it is for \((X_1, X_2)\) to be near \((x_1, x_2)\). 

\((*)\) \(\mathbb{P}(X_1 = x_1, X_2 = x_2) = 0\). 
\(\mathbb{P}(X_1 = x_1, X_2 \in A_2) = 0\). 

Also \(\mathbb{P}(X_1 = X_2) = 0\) since a "line" has no area.
Like in the discrete case, the marginals of the joint p.d.f. give the p.d.f.'s of $X_1$ and $X_2$:

$$P(X_1 \leq x_1) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x_1, x_2) \, dx_2 \right) \, dx_1$$

$\Rightarrow$ p.d.f. of $X_1$ is $f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) \, dx_2$

Similarly, p.d.f. of $X_2$ is $f_{X_2}(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2) \, dx_1$

Two cont. r.v. don't necessarily have a joint p.d.f! If they do, we call them jointly continuous.
Independence of random variables:

Recall: events $A, B$ are independent if $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$.

and Two r.v. $X_1$ and $X_2$ are independent if

\[ \Pr(X_1 \in A_1, X_2 \in A_2) = \Pr(X_1 \in A_1) \cdot \Pr(X_2 \in A_2) \]

for all $A_1, A_2 \subset \Omega$. defined on the same sample space.
For discrete r.v., this is the same as

$$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1) \cdot P(X_2 = x_2)$$

joint p.m.f.

product of individual p.m.f.'s

For jointly cont. r.v., this is the same as

$$f(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

joint p.d.f.

product of individual p.d.f.'s

for all $x_1, x_2 \in \mathbb{R}$. 
In the previous example, X and Y were independent!

Example: Choose a random uniform point on the unit disk. Call the coordinates \((X_1, X_2)\). Are \(X_1\) and \(X_2\) indep?

Remark: By a uniform point in a region \(A \subset \mathbb{R}^2\), we mean that the coordinates \((X_1, X_2)\) have joint p.d.f. which is constant on \(A\) and 0 outside \(A\).
\textbf{Sol.:} joint p.d.f. is \( f(x_1, x_2) = \begin{cases} \frac{1}{\pi} & \text{if } x_1^2 + x_2^2 \leq 1 \\ 0 & \text{otherwise} \end{cases} \)

We have to check whether

\[ f(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2). \]

We have

\[ f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) \, dx_2 = \begin{cases} \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \frac{1}{\pi} \, dx_2 & \text{if } |x_1| \leq 1 \\ 0 & \text{if } |x_1| > 1 \end{cases} \]

\[ = \begin{cases} \frac{2}{\pi} \sqrt{1-x_1^2} & \text{if } |x_1| \leq 1 \\ 0 & \text{otherwise} \end{cases} \]
Similarly, 

\[ f_{X_2}(x_2) = \begin{cases} \frac{2}{\pi} \sqrt{1-x_2^2} & \text{if } |x_2| \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

Clearly, 

\[ f(x_1, x_2) \neq f_{X_1}(x_1) \cdot f_{X_2}(x_2) \quad \text{for all } x_1, x_2. \]

\[ \Rightarrow \quad X_1 \text{ and } X_2 \text{ are not indep.} \]

\underline{Example:} \quad \text{Let } X_1, X_2 \text{ be chosen independently and uniformly in } [0, 2].

What is the prob. that their distance is less than 1?
Sol: The joint p.d.f. is \( f(x_1, x_2) = \begin{cases} \frac{1}{4} & \text{if } 0 < x_1, x_2 < 2 \\ 0 & \text{otherwise} \end{cases} \)

\[ = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \]

\[ \Rightarrow \mathbb{P}( |X_1 - X_2| < 1) = \]

\[ = \mathbb{P}( (X_1, X_2) \in A) \]

\[ = \frac{3}{4} \cdot |A| = \left[ \frac{3}{4} \right]. \]

\[ \text{area of } A \]

\[ \frac{\iint_A \frac{1}{4} \, dx_1 \, dx_2}{A} = \frac{1}{\frac{1}{4}} \int \int A dx_1 dx_2 = \frac{1}{4} \cdot |A| \]