Correction: \[ P(X > \frac{1}{2}) = 1 - P(X \leq \frac{1}{2}) = 1 - \Phi(\frac{1}{2}). \]

Today: Joint distributions

Idea: How to compute probabilities defined in terms of more than one r.v. (defined on the same sample space).

We begin with the discrete case:

Example: Consider an urn with 3 white balls and 2 black balls.

Draw all five, one-by-one.

Let \( X \) = \# of draws until first black ball.
Let \( Y \) = \# of draws between first and second black balls.

E.g. BWBWB
\( X = 1 \)
\( Y = 3 \)
The joint pmf of $X$ and $Y$ is given by

$$p(x,y) = P(X=x, Y=y)$$

over the possible values of $X$ and $Y$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{10}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{10}$</td>
<td>$\frac{1}{10}$</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{10}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Let's compute the joint pmf for this example. We can make a table for this.
\[ \sum_{x,y} p(x,y) = 1 \]

**Normalization:**

\[ p(x,y) = 0 \quad \text{for all} \quad x \neq x' \]

**Positivity:**

There are 10 such businesses.

\[ \text{Why } \frac{1}{2} \text{? Each pair } (x,y) \text{ describes a} \]

**Explanation:** Why is \( x + y > 5 \) impossible?
The row sums of the joint pmf give the pmf of $X$:

$$p_x(x) = \sum_y p(x, y)$$

This is called the marginal distribution of $X$.

The column sums of the joint pmf give the pmf of $Y$:

$$p_y(y) = \sum_x p(x, y)$$

This is the marginal distribution of $Y$. 

\[ \text{Eq.} \]
Independence:

Recall that $X$ and $Y$ are independent if

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$$

for any $A, B \subset \mathbb{B}$.

**Fact:** For discrete r.v., this is the same as

$$p(x, y) = p_X(x) \cdot p_Y(y)$$

> joint pmf

> individual pmfs of $X$ and $Y$. 
In the example, $X$ and $Y$ are not independent since $p(x, y) = 0 \neq p(x) \cdot p(y)$. Are $X$ and $Y$ independent in the following?

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y = 1$</th>
<th>$Y = 2$</th>
<th>$Y = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
</tr>
<tr>
<td>2</td>
<td>1/6</td>
<td>1/6</td>
<td>1/12</td>
</tr>
<tr>
<td>3</td>
<td>1/12</td>
<td>1/2</td>
<td>1/12</td>
</tr>
</tbody>
</table>

Yes!
Transformations: Given \( g: \mathbb{R}^2 \rightarrow \mathbb{R} \), we want to understand the r.v. \( g(X,Y) \).

To find the pmf of \( Z = g(X,Y) \):

\[
P_Z(z) = P(g(X,Y) = z) = \sum_{(x,y) \text{ such that } g(x,y) = z} P(x,y)
\]

Example: Roll 3 dice.

\[X = \text{sum}, \ Y = \text{product}.
\]

\[
P_X(4) = P(X=4) = P(\text{two 1's, one 2}) = 3 \cdot \left( \frac{1}{6} \right)^3
\]
\[
P_X(3) = P(Y=1) = P(\text{all 1's}) = \left( \frac{1}{6} \right)^3.
\]
\[
P_X(17,180) = P(X=17, Y=180) = P(\text{two 6's, one 5}) = 3 \cdot \left( \frac{1}{6} \right)^3.
\]