Expectation / average value of a r.v.

Motivation: You have the option of playing a gambling game with the following rules.
A fair die is rolled:
- even number $\rightarrow$ win amount on the die in $\$.
- odd number $\rightarrow$ lose $\$3$.

Q: Is the game in your favor?

$\rightarrow$ Intuitively, the average winnings (profit) in such a game is the total winnings after a large number of rounds played, divided by the number of games played.
In this game:
- Die is odd \(\rightarrow\) 50% of the time, lose $3
- Die is 2 \(\rightarrow\) 1/6 of the time, gain $2
- Die is 4 \(\rightarrow\) 1/6
- Die is 6 \(\rightarrow\) 1/6

So after many games, say \(N\), we can "expect" to win roughly:
\[ \frac{N}{2} \cdot (-3) + \frac{N}{6} \cdot 2 + \frac{N}{6} \cdot 4 + \frac{N}{6} \cdot 6 = \frac{N}{2} \]

\[ \Rightarrow \quad \text{That is, per game played, we expect to win } \frac{1}{2}. \]
For a given point $a$, the possible values are:

$X = \{ 2 \times P(x) \}

\text{The expectation/mean/average value of } X \text{ whose point is } p(x) \text{ is def.}

\text{It is what we expect, not what actually happens. The expected winnings is not a random number.}

\text{That would be $-3$). Of course, this is not the most probable amount to win per game (it would be $-2$.}
Examples:

(I) Expectation of $\text{Ber}(p)$ r.v. $X$:

- pmf $p(0) = 1-p$, $p(1) = p$.
- $E[X] = 0 \cdot (1-p) + 1 \cdot p = p$.

(II) $X \sim \text{Unif}(\{1, 2, \ldots, n\})$

- pmf $p(k) = \frac{1}{n}$ for all $k = 1 \ldots n$.
- $E[X] = \sum_{k=1}^{n} k \cdot \frac{1}{n} = \frac{1}{n} \left(1 + 2 + 3 + \ldots + n\right) = \frac{n+1}{2}$

e.g. Roll a die. The expected value is $3\frac{1}{2}$.
\[
\text{pmf } p(n) = e^{-n} \frac{n^k}{k!}, \quad k = 0, 1, 2, \ldots \n
\text{pmf } p(k) = \left( \frac{n}{k} \right) p(k)(1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n
\]

\[
\text{np}(p+1-p)^{n-1} = np
\]

\[
\text{Roll a die 30 times, the expected number of "3s" is 15}
\]

\[
\text{EX } = \sum_{n=0}^{\infty} n \cdot p(n) = \sum_{n=0}^{\infty} n \cdot e^{-n} \frac{n^k}{k!}
\]

\[
\text{EX } = \sum_{k=0}^{\infty} k \cdot \left( \frac{n}{k} \right) p(k)(1-p)^{n-k} = np
\]
(IV) \( X \sim \text{Geom}(p) \), pmf \( p(k) = p \cdot (1-p)^{k-1} \), \( k = 1, 2, \ldots \)

e.g. The expected number of die rolls until getting a "3" is

\[ 6 = \frac{1}{1/6} \rightarrow \frac{1}{p}. \]

\[ \text{EX} = \sum_{k=1}^{\infty} k \cdot p \cdot (1-p)^{k-1} \]

\[ = p \cdot \frac{\left( \frac{1}{1-x} \right)'}{(1-x)^2} \bigg|_{x=1-p} \]

\[ = p \cdot \frac{1}{(1-x)^2} \bigg|_{x=1-p} = p \cdot \frac{1}{p^2} = \frac{1}{p}. \]
Often, instead of expectations of r.v. $X$, we will also be interested in functions of $X$, e.g. $X^2$, $\log X$, $\sqrt{X}$, ...

**Example:** Roll a die. Let $X$ be the outcome. Let $g(x) = (x-3)^2$. What is $\mathbb{E}g(X)$?

**Note:** $g(X)$ is a r.v. itself.

So $\mathbb{E}g(X) = \sum_a a \cdot \mathbb{P}(g(X) = a)$

$$= 1 \cdot \frac{2}{6} + 1 \cdot \frac{2}{6} + 0 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6}$$

$X=1$ or $5$ $X=2$ or $4$ $X=3$ $X=6$