Continuous r.v., cdf/pdf: $f(x) = \begin{cases} \frac{1}{a} & x < 0 \\ 0 & x \geq 0 \end{cases}$

Exponential with rate $\lambda$: $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

Uniform on $[a,b]$: $f(x) = 1$ for $x \in [a,b]$, otherwise $0$.

Last time: (4)

Example:

- You run a taco place downtown. There are probably thousands of people on the street, but it's 2pm and they've already eaten. You're tired and decide to take a 20-minute break. You know from experience that you will have 2 customers while you nap. What is the prob.
We want \( P(\text{time of next arrival} > 2:20 \text{ pm}) = \)

\[ = P(\text{time elapsed until next arrival} > \frac{1}{3} \text{ hour}) = 0 \]

→ Can split the hour interval into three 20-minute intervals.

Prob. of an arrival in each interval is \( p = \frac{2}{3} \) (since \( 3p = 2 \) walkins per hour)

\[ = P(\text{time elapsed} > \frac{1}{3}) = P(\text{no arrival in first interval}) = 1 - p = \frac{1}{3} \]

Assuming that at most one person arrives in any interval.
Now split into six 10-min intervals. Now $p = \frac{1}{3}$.

\[(6p = 2)\]

\[\Omega \cong P(\text{no arrival in first two intervals}) = (1-p)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}\]

Now split into $n$ intervals of length $\frac{1}{n}$ each.

Here $p = \frac{2}{n}$ (np = 2).

\[\Omega \cong P(\text{no arrival in the first } \frac{n}{3} \text{ intervals}) = P(\text{Bin}\left(\frac{n}{3}, \frac{2}{n}\right) = 0)\]

\[\Omega = P(\text{first interval of arrival is } > \frac{n}{3}) = P(\text{Geom}\left(\frac{2}{n}\right) > \frac{n}{3})\]
\[
\lim_{n \to \infty} (1 - \frac{2}{n})^{n/3} = \quad?
\]
\[
\lim_{n \to \infty} \text{P}(\text{Poisson}(\frac{2}{3}) = 0) = e^{-\frac{2}{3}} \cdot \frac{\left(\frac{2}{3}\right)^0}{0!} = e^{-\frac{2}{3}}
\]

Similarly,
\[
\text{P}(\text{time elapsed until } T > t) = e^{-2t}
\]
\[
\Rightarrow \quad F_X(t) = \text{P}(X \leq t) = \begin{cases} 
1 - e^{-2t} & \text{if } t \geq 0 \\
0 & \text{else}
\end{cases}
\]
\[
\Rightarrow \quad \text{So, } X \sim \text{Exp}(2).
\]
So the exponential r.v. is the continuous analogue of the geometric r.v., in that it measures how long it takes until the first arrival.

Memoryless property of exponential r.v.:

\[ X \sim \text{Exp}(\lambda) \]

\[ P(X > s + t \mid X > t) = P(X > s) \]

\[ \frac{P(X > s + t)}{P(X > t)} = \frac{1 - P(X \leq s + t)}{1 - P(X \leq t)} = \frac{e^{-\lambda(s + t)}}{e^{-\lambda t}} = e^{-\lambda s} \]
Normal / Gaussian r.v.

Standard normal r.v. \( X \sim N(0, 1) \)

pdf: \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \)
Check normalization:

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = 1, \quad I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \]

Need to show \( I = \sqrt{\frac{\pi}{2}} \).

Trick: Show instead that \( I^2 = 2\pi \).

\[
I^2 = \int_{-\infty}^{\infty} e^{-x^2} \, dx \cdot \int_{-\infty}^{\infty} e^{-y^2} \, dy = \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2} \, dx \, dy
\]

Using polar coordinates:

\[ x = r \cos \theta, \quad y = r \sin \theta \]

\[ x^2 + y^2 = r^2 \]

\[ dxdy = r \, dr \, d\theta \]

\[
= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2/2} \cdot r \, dr \, d\theta
= 2\pi \int_{0}^{\infty} r^2 e^{-r^2/2} \, dr
= 2\pi \cdot \left( -e^{-r^2/2} \right) \bigg|_{0}^{\infty} = 2\pi.
\]
$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \Phi(x)$

$P(\mid X \mid < 1) = P(-1 < X < 1) = P(X < 1) - P(X \leq -1) = \Phi(1) - \Phi(-1) = \frac{1}{2} \phi(1) - \frac{1}{2} \phi(-1) = \frac{1}{2} \phi(1) - \frac{1}{2} \phi(-1) = \frac{1}{2} (\phi(1) - \phi(-1))$

$P(X = 1) = P(X = -1) = 1 - 0.84134 - 1 = 0.15866$

$P(X = 0) = 1 - P(X = 1) - P(X = -1) = 1 - 0.84134 - 0.15866 = 0.00000$
Table 1: Values of \( \Phi(z) = \int_{-\infty}^{z} e^{-\frac{x^2}{2}} \, dx \), where \( f(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} \) is the density of \( \mathcal{N}(0, 1) \) random variable.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( \Phi(z) )</th>
</tr>
</thead>
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</tr>
<tr>
<td>1.0</td>
<td>0.99999</td>
</tr>
</tbody>
</table>

Note: The table values are rounded to four decimal places for simplicity.