Rule: (law of total prob.)

For any two events $E, F$:

$$P(E) = P(E|F) \cdot P(F) + P(E|F^c) \cdot P(F^c)$$

Proof:

$$P(E) = P((E \cap F) \cup (E \cap F^c))$$

$$= P(E \cap F) + P(E \cap F^c)$$

$$= P(E|F) \cdot P(F) + P(E|F^c) \cdot P(F^c)$$

axiom III

Diagram: Two events $E$ and $F$ with $E \cap F$ shaded.
Remark: the pair of events $F, F^c$ constitutes a partition of $S$.

i.e., $F \cup F^c = S$, $F \cap F^c = \emptyset$.

General law of total prob.

Let $E$ be an event and $F_1, F_2, \ldots, F_n$ a partition of $S$.

Then

$$P(E) = \sum_{k=1}^{n} P(E | F_k) P(F_k)$$
Back to the example:

Choose one at random:
- Cube \( \rightarrow \) 6-sided
- Tetrahedron \( \rightarrow \) 4-sided

\[ P(\text{roll } 2) = P(\text{roll } 2 \mid \text{tetra}) P(\text{tetra}) + P(\text{roll } 2 \mid \text{cube}) P(\text{cube}) \]

\[ = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2} = \frac{5}{24} \]
If you roll a 2, what is the probability that you used the tetrahedron?

\[ P(\text{tetra and roll 2}) = \frac{P(\text{roll 2})}{P(\text{roll 2})} \]

\[ = \frac{\frac{1}{6}}{\frac{1}{2}} \]

\[ = \frac{1}{3} \]

\[ = \frac{3}{5} \]

We saw this last time.
(Bayes's formula)

Rule: For any two events $E, F$:

$$P(F|E) = \frac{P(E|F) \cdot P(F)}{P(E)} \quad \text{or} \quad P(E|F) = \frac{P(F|E) \cdot P(F)}{P(E)}$$

Interpretation:

We have a two-step experiment. E.g., stage 1: randomly choose a dice. Stage 2: roll the chosen dice.
→ in stage 1, probabilities are known.

→ in stage 2, conditional probabilities given the outcome of the first stage are known.

Baye's formula tells us how to compute conditional probabilities of stage 1, given the outcome of stage 2.

**General Bayes formula:**

Let $E$ be an event and $F_1, ..., F_n$ a partition of $S$.

Then $P(F_i | E) = \frac{P(E | F_i) \cdot P(F_i)}{P(E)} = \frac{\sum_{k=1}^{n} P(E | F_k) \cdot P(F_k)}{P(E)}$
Exercise:

Two countries: A and B.

If you're born in A, you have a 90% chance of being rich. If you're born in B, you have a 1% chance of being rich. Given that you're rich, what are the chances that you come from A? (Same information is missing.)
The population of A: 8 Mil
B: 1.2 Bil

We first need to know:

- \( P(\text{rich in A}) = \frac{8}{1200} = 0.0067 \)
- \( P(\text{rich in B}) = \frac{1000}{1200} = 0.8333 \)

From the problem data:

- \( P(\text{A | rich}) = 0.9 \)
- \( P(\text{B | rich}) = 0.1 \)

Bayes:

\[
P(\text{rich | A}) = \frac{P(\text{rich}) \cdot P(\text{A | rich})}{P(\text{rich})} = \frac{0.9 \cdot \frac{8}{1200}}{0.9 \cdot \frac{8}{1200} + 0.1 \cdot \frac{800}{1200}}
\]

\[
P(\text{rich | A}) = \frac{9}{37.5} = 0.24
\]

\[
P(\text{rich}) = \frac{3}{8}
\]

\[
P(\text{rich | A}) = \frac{3}{8} = \frac{3}{8}
\]