Last time:

Condition prob:

\[ P(E|F) = \frac{P(\text{EnF})}{P(F)} \]

Independence:

\[ P(E|F) = P(E) \]

Example:

Flip a coin 3 times.

\[ E = \text{"exactly one of flips \#1 and \#2 is Heads"} \]
\[ F = \text{"\#2 and \#3 is Heads"} \]
\[ G = \text{"\#1 and \#3 is H"} \]

Each event has prob. \( \frac{1}{2} \).

\( \square \) Are \( E \) and \( F \) indep?

\[ P(E\cap F) = P(\{\text{HTH, THT}\}) = \frac{2}{8} = \frac{1}{4} \]

So \[ P(E\cap F) = P(E) \cdot P(F) \Rightarrow E \text{ and } F \text{ are indep!} \]
Similarly, $F$ and $G$ are indep.

and, $E$ and $G$ are indep.

Should we say that $E, F, G$ are indep?

NO! because $G$ cannot occur if both $E$ and $F$ occur.

That is, $E \cap F \cap G = \emptyset$

$\Rightarrow E \cap F$ and $G$ are not indep!

$0 = P((E \cap F) \cap G) + P(E \cap F) \cdot P(G) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$

Check: $E, F \cup G$ indep?
Def: A collection of events $E_1, E_2, \ldots, E_n$ is called indep (mutually indep) if the prob. of any intersection is equal to the product of the individual probabilities:

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \cdot P(E_{i_2}) \cdots P(E_{i_k})$$

An equivalent definition: $E_1, \ldots, E_n$ are indep. if and only if

$$P(F_1 \cap \cdots \cap F_n) = P(F_1) \cdots P(F_n)$$

For example:

$$P(E_1 E_2 E_3^c E_4 E_5 E_6^c) = P(E_1) \cdots P(E_6)$$
Caution: As the example showed, independence of \( n \) events is not the same as independence of every subset of \( n-1 \) events.

Example:

(1) Roll five dice: \( E_1 = "\text{first die is 3}" \)
\( E_2 = "\text{second die is even}" \)
\( \ldots \)
\( E_5 = "\text{fifth die is \ldots}" \)

\( \Rightarrow \) \( E_1, \ldots, E_5 \) are indep.
(II) Remove five balls from an urn (balls are numbered)

\[ E_1 = \text{"first is ..."} \]
\[ E_2 = \text{"second is ..."} \]
\[ \vdots \]
\[ E_5 = \text{"fifth is ..."} \]

\[ \Rightarrow E_1, \ldots, E_5 \text{ are dependent in general.} \]

Law of total probability (and Bayes's formula)

Example: You have 2 dice: one is a standard "cube" die (6 sides) and one is a tetrahedron (4 sides). You randomly select one of them and roll it.
What is the prob. of rolling 2?
If you did roll a 2, what is the prob. that you used the tetrahedron.

Solution: sample space: \( S = \{a_1, \ldots, a_6, b_1, \ldots, b_4\} \)

\[ |S| = 10 \]

\[ \text{cube} \quad \text{tetrahedron} \]

\[ \Rightarrow \text{uniform prob.? NO!} \]

Let's compute the probabilities:
\[ F = \text{"rolled the tetrahedron"} = \{6_1, 6_2, 6_3, 6_4, 5\} \]
\[ F^c = \text{"rolled the cube"} = \{a_1, ..., a_6\} \]

By the description of the problem:

\[ \mathbb{P}(F) = \frac{1}{2} = \mathbb{P}(F^c) \]

Define events \( E_1, ..., E_6 \):

\[ E_i = \text{"rolled i"} \quad \rightarrow \quad E_i = \{a_i, b_i, 3\} \text{ if } 1 \leq i \leq 4 \]
\[ E_i = \{a_i, 3\} \text{ if } i = 5, 6 \]

Note that individual outcomes can be expressed as intersections:

\[ E_2 \cap F = \{6_2, 3\} \]
\[ E_2 \cap F^c = \{a_2\} \]
From the problem description:

\[ P(E_1 | F) = \ldots = P(E_{10} | F) = \frac{1}{16} \]

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\[ B(2) \] rolled 2 using tetrahedron

\[ = P(E_2 | F) \cdot P(F) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} \]
\[
P(\text{rolled 2 using cube}) = P(2,1|F) = P(E_2|F) = 1/6 \\
\Rightarrow P(\text{rolled 2 using die}) = 1/12
\]