Last time: Conditional probability

\[ P(E \mid F) = \frac{P(E \cap F)}{P(F)} \]

the prob. of \( E \) given \( F \)

Example: Roll two dice.

\[ E = \{ \text{sum is 5} \} \quad F = \{ \text{one die is 3} \} \quad (\text{at least}) \]

\[ P(E) = \frac{4}{36} = \frac{1}{9} \]

\[ P(F) = \frac{11}{36} \]

\[ P(E \cap F) = \frac{2}{36} = \frac{1}{18} \]

\[ P(E \mid F) = \frac{\frac{1}{18}}{\frac{11}{36}} = \frac{2}{11} \]

\[ P(F \mid E) = \frac{\frac{1}{18}}{\frac{1}{9}} = \frac{1}{2} \]
Remark: Often we already know $P(E|F)$ in some other way, and then it is useful that $P(E|NF) = P(E|F) \cdot P(F) = P(F|IE) \cdot P(E)$.

Example: Draw 2 cards from a standard deck. (one after another)

What is $P(\text{both cards are kings})$?

Solution:

\[
\frac{\binom{4}{2}}{\binom{52}{2}} \quad \leftarrow \text{choose two keys}
\]
Sol II: Imagine cards are drawn consecutively.

F = "first card is a King"
E = "second card is a King"
ENF = "both cards are kings"

\[ P(F) = \frac{4}{52} = \frac{1}{13} \]
\[ P(E | F) = \frac{3}{51} = \frac{1}{17} \]
\[ P(ENF) = P(F) \cdot P(E | F) = \frac{1}{13} \cdot \frac{1}{17} = \frac{4}{52} \]

choose a king from a deck of 51 having only 3 kings.
Fact: (verify this yourself)

Conditional probabilities satisfy the same axioms/properties of a "normal" probability.

(I) \[ 0 \leq P(E|F) \leq 1 \]

(II) \[ P(S|F) = 1 \]

(III) \[ P(E_1 \cup E_2 | F) = P(E_1|F) + P(E_2|F) \]

whenever \( E_1 \) and \( E_2 \) are disjoint.

(IV) \[ P(E^c|F) = 1 - P(E|F) \]

(V) \[ P(E_1 \cup E_2 | F) = P(E_1|F) + P(E_2|F) - P(E_1 \cap E_2 | F) \]
\[ P(E \cup F) = P(E) + P(F) \]

\[ P(E \cup F_1 \cup F_2) = P(E) + P(F_1) + P(F_2) \]

Even if \( F_1, F_2 \) are disjoint, \( P(E \cup F_1 \cup F_2) = 1 \)

Two events \( E \) and \( F \) are called independent if

\[ P(E \cap F) = P(E) \cdot P(F) \]

Caution:

\[ E = S \implies P(E \cap F) = 0 \]

Def:

\[ P(E | F) = \frac{P(E \cap F)}{P(F)} \]

\[ P(F | E) = \frac{P(E \cap F)}{P(E)} \]
Geometric interpretation:

Example:

\[ P(E) = \frac{1}{2} \]
\[ P(F) = \frac{1}{2} \]
\[ P(ENF) = \frac{1}{4} \]

\[ \rightarrow E \text{ and } F \text{ are indep!} \]
Draw a single card from a standard deck.

- E: "it's a king"
- F: "it's spades"

\[
P(E) = \frac{13}{52} = \frac{1}{4} \quad P(F) = \frac{13}{52} = \frac{1}{4}
\]

\[
P(E \cap F) = \frac{13}{52} = \frac{1}{4}
\]

E and F are independent.

Example 1:

Check:
Draw two cards. E = "first is King", F = "second is King."

\[ P(E) = \frac{3}{52} \]
\[ P(E|F) = \frac{1}{13} \]
\[ P(F|E) = \frac{1}{51} \]

We've already seen that

\[ \frac{P(E|F)}{P(F)} = \frac{1}{13} \]
\[ \frac{P(F|E)}{P(E)} = \frac{1}{52} \]

So \( E \) and \( F \) are not indep.

Example II:

\[ P(E|F) \neq P(E) \]

\[ \Rightarrow \]