Non-uniform probabilities and introduction to random variables

Uniform prob : \( P(E) = \frac{|E|}{|S|} \)

Example: Sample space \( S = \{1, 2, 3, 4, \ldots, \infty\} \)

Toss a coin until the first time you get Heads, and record the number of flips.

\[ P(\text{first H is on } n\text{'th toss}) = \]
\[ = P(\text{when tossing } n \text{ coins, get } TTT\ldots TT H) = \frac{1}{2^n} \]

\[ \Rightarrow P(\{n\}) = 2^{-n} \text{ for } n=1, 2, 3, \ldots \]

\[ P(\{oo3\}) = ? \]
\[ P(N) = \sum_{n=1}^{\infty} P(3n) = \sum_{n=1}^{\infty} 2^{-n} = ? \]

\[ \text{e.g. } P(\{1,2,3\}) = P(\{13\}) + P(\{23\}) + P(\{53\}) \]

\textbf{Claim:} \[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x|<1 \]

\[ y \equiv 1 + x + x^2 + x^3 + \ldots \]

\[ xy = x + x^2 + x^3 + \ldots = y - 1 \]

\[ 1 = y(1-x) \]

\[ y = \frac{1}{1-x} \]

\[ P(N) = \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n = \frac{1}{2} \cdot \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \]
\[ P(S) = P(N \cup \{oo3\}) = P(N) + P(\{oo3\}) \]

But \[ P(N) = 1 = P(S) \]

So \[ P(\{oo3\}) = 0. \]

Example: uniform letter from this sentence.

Sample space \( S = \{ u, n, i, t, o, r, m, l, e, t, h, s, c \} \).

\[ |S| = 13 \]

\[ P(\{it3\}) = \frac{4}{29} \]

\[ P(\{oo3\}) = \frac{1}{29} \]

\[ P(\{ro,i,e,ug\}) = \frac{10}{29} \]
Random variables - intro:

Def: A random variable (r.v.) is a function from the sample space to the real numbers $\mathbb{R}$. That is, we associate to each outcome a number.

E.g. In the uniform letter example, let $X$ be the "value" of the letter in Scrabble.

Here: $M, C \Rightarrow 3$
$H, F \Rightarrow 4$
rest \Rightarrow 1$

This gives a new sample space $S = \{1, 3, 4, 8\}$
Notation: For r.v.'s, we use letters like $X, Y, Z, \ldots$ instead of $f, g, h, \ldots$

So $X: S \rightarrow \mathbb{R}$. 

We think of $X$ as the random number $X(s)$, where $s$ is the outcome of the experiment.

(4) Discrete r.v.: having its values in a sequence $x_1, x_2, x_3, \ldots$.

(5) Continuous r.v.: continuum possible values e.g. $[0, 1]$. 
We write \( \{ X \in A \} = \{ s \in S : X(s) \in A \} \)

\[ \text{e.g. } \mathbb{P}(X = s) = \mathbb{P}(\{ s \in S : X(s) = s \}) \]

**Discrete r.v.:** If the values are \( x_1, x_2, x_3, \ldots \),
then \( \mathbb{P}(X = x_i) \) is some number.

Denote \( p_a = \mathbb{P}(X = a) \)

\[ \sum_a p_a = 1 \]

\[ \mathbb{P}(X \in \{ x_1, x_2, x_3, \ldots \}) = 1 \]

\[ p_{x_1} + p_{x_2} + p_{x_3} + \ldots \]
Recall the example of tossing a coin until first heads.

\[ X = \{1, 2, 3, \ldots \} \]

\[ P(X = n) = 2^{-n} \]

\[ p_n = \sum_{n=1}^{\infty} P(X = n) = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32} - \frac{1}{64} - \frac{1}{128} - \frac{1}{256} - \ldots \]

\[ p_n = 2(1 - \frac{1}{2}) = 1 - \frac{1}{2} \]

Time to get first 6 when tossing a die.

\[ X = \{1, 2, 3, 4, 5, 6\} \]

\[ P(X = n) = \frac{1}{6} \]

\[ p_n = \sum_{n=1}^{6} P(X = n) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1 \]

Example:

\[ p_n = 2 \]