In how many ways can you choose 7 balls from a bin with 3 red and 3 blue balls (unlimited supply of each)?

Order doesn't matter.

Solution:

(1) All same color

(a) $\text{aaa}$

(b) $\text{aab}$

(2) All 3 colors

$\text{abc}$

(3) Exactly two colors

$\text{aab}$

$\text{abb}$

Total: $3 + 3 + 6 + 3 = 15$
<table>
<thead>
<tr>
<th>Rule: $#$ ways to choose $k$ elements from a set of size $n$</th>
</tr>
</thead>
<tbody>
<tr>
<td># of $k$ elements with replacement is $\binom{n+k-1}{k}$</td>
</tr>
<tr>
<td>$\frac{n!}{(n-k)!}$ with replacement</td>
</tr>
<tr>
<td>$\frac{n!}{(n-k)!} \binom{n}{k}$ ordered</td>
</tr>
<tr>
<td>$\frac{n!}{(n-k)!}$ unordered</td>
</tr>
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Summary:

- If you choose $k$ elements from a set of size $n$, it is:
  - $\binom{n+k-1}{k}$ with replacement
  - $\frac{n!}{(n-k)!}$ ordered
  - $\frac{n!}{(n-k)!}$ unordered
Binomial Theorem:

\[(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\]

**Example:**

\[(x+y)^3 = \binom{3}{0} x^3 y^0 + \binom{3}{1} x^2 y^1 + \binom{3}{2} x^1 y^2 + \binom{3}{3} x^0 y^3\]

**Explanation:**

To get the terms in the expansion, we use the binomial coefficients from binomial expansion, with the following possibilities:

- \(x^3 y^0 = x^3\) from 3 terms,
- \(x^2 y = \binom{3}{1} x^2 y\) from \(\binom{3}{1}\) such possibilities,
- \(x y^2 = \binom{3}{2} x y^2\) from \(\binom{3}{2}\) such possibilities,
- \(y^3 = \binom{3}{3} y^3\) from \(\binom{3}{3}\) such possibilities.
(Including the empty set and the entire set)

How many subsets of \(1, 2, \ldots, n\) are there?

Let the number of subsets of size \(k\) be \(\binom{n}{k}\).

\[ \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} + \binom{n-1}{k-2} + \cdots + \binom{n-1}{0} \]

By the binomial theorem with \(x = y = 1\),

\[ (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \]

Another solution:

Total # of subsets = \(\sum_{k=0}^{n} \binom{n}{k}\)

For each element \(i \in \{1, 2, \ldots, n\}\),

decide if to include it or not.

\[ 2 \times 2 \times \cdots \times 2 = 2^n \]
Example: Every day, Mon-Sun, given a choice of 1 of 3 fruits: apple, orange, banana.

A "healthy menu" consists of 3 apples, 2 oranges, 2 bananas.

How many healthy weekly menus can we create?

Sol:

\[
\begin{align*}
\text{MTWTS} & \text{SS} \\
\odot \odot \odot \odot \odot \odot \\
\binom{7}{2} \cdot \binom{5}{3} \cdot \binom{2}{2} &= 21 \cdot 10 \cdot 1 = 210
\end{align*}
\]

Choose days for bananas, apples, oranges.
More generally: given $n$ days, $n_i$ of fruit "A"
$n_2$ of fruit "B"
\[ \vdots \]
$n_m$ of fruit "Z"

\[ n = n_1 + n_2 + \ldots + n_m \]

# healthy menus

\[ \frac{n!}{n_1! (n-n_1)!} \cdot \frac{(n-n_1)!}{n_2! (n-n_1-n_2)!} \cdot \frac{(n-n_1-n_2)!}{n_3! (n-n_1-n_2-n_3)!} \cdot \ldots \cdot \frac{(n-n_1-n_2-\ldots-n_{m-1})!}{n_m!} \]

\[ \frac{n!}{n_1! n_2! \ldots n_m!} \]