Math 302, assignment 8

1. The random variables $X, Y$ have joint probability density function

$$f(x, y) = \begin{cases} Cye^{-y-x/y} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of $C$? Hint: Integrate with respect to $x$ first.
(b) Find the marginal probability density function $f_Y$.
(c) Are $X$ and $Y$ independent?
(d) Compute $P(X \leq Y^2)$.

2. Let $X \sim \text{Exp}(1/2)$ and $Y \sim \text{Unif}([2,4])$ be independent. Calculate $P(Y - X \geq 1/2)$.

3. Let $X \sim \text{Exp}(\lambda_1)$ and $Y \sim \text{Exp}(\lambda_2)$ be independent. Find the following:
   (a) $P(X < Y)$.
   (b) distribution of $\min(X, Y)$.
   (c) distribution of $X + Y$.

4. Suppose a laser pointer is located at the origin of your coordinate system, and is pointing toward the vertical line $L = \{(x, y) : x = 1\}$. Suppose that the angle $X$ between the laser beam and the $x$ axis is a uniform random variable. Calculate the p.d.f. of the $Y$ coordinate of the point on $L$ which the beam points at. Hint: Draw a picture and argue that this means that $X \sim \text{Unif}[-\pi/2, \pi/2]$ and $Y = \tan X$.

5. Let $X$ and $Y$ be two random variables.
   (a) Show that $\text{cov}(X, Y) = E[(X - E(X))(Y - E(Y))]$.
   (b) Show that $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{cov}(X, Y)$.

6. Let $X \sim \text{Unif}[-1,1]$. Show that the random variables $X$ and $X^2$ are uncorrelated, but not independent.

7. Suppose that $X, Y$ are discrete random variables with joint p.m.f. given by

<table>
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<th>$X \downarrow Y \rightarrow$</th>
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<th>2</th>
<th>3</th>
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(a) Calculate $\text{cov}(X, Y)$.
(b) Calculate $\text{Var}(X + Y)$.
(c) Calculate $\text{cov}(U, V)$, where $U = 2X + Y$ and $V = 2X - Y$.

Extra practice problems (do not hand in): Chapter 6, problems 6,8,12,28,29.