1. Let $X$ take values in $\{1, 2, 3, 4, 5\}$ and have p.m.f. given by

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = k)$</td>
<td>$1/7$</td>
<td>$1/14$</td>
<td>$3/14$</td>
<td>$2/7$</td>
<td>$2/7$</td>
</tr>
</tbody>
</table>

(a) Calculate $E[X]$  
(b) Calculate $E[X − 2]$  
(c) Calculate $\text{Var}(X)$

2. A random variable has p.d.f.

$$f(x) = \begin{cases} 
e^x & 0 \leq x \leq a, \\ 0 & \text{otherwise}, \end{cases}$$

where $a > 0$ is some number.

(a) Find the value of $a$.  
(b) Compute $E[X]$.  
(c) Compute $\text{Var}(X)$.

3. A continuous RV $X$ has the p.d.f.

$$f(x) = \begin{cases} \frac{c}{1+x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(a) Find $c$.  
(b) Compute $E[X]$.  
(c) Compute $E\left(\frac{1}{\sqrt{1+X^2}}\right)$.

4. An urn contains 5 black and 5 white balls. You draw balls one by one without replacement, until you draw your first white ball. The total number of balls you have drawn, including the first white ball, is a random variable $X$.

(a) Calculate the p.m.f. of $X$.  
(b) Calculate the expectation $E[X]$.  
(c) Calculate the variance $\text{Var}(X)$.

5. Let $X$ be a random variable (discrete or continuous). Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = E(X^2)$$.  

Show that $f$ attains its minimum at $t = E[X]$.

6. A fair coin is tossed 50 times. The outcomes are written in order, producing a 50-letter “word” consisting of the letters H and T. Compute the expected number of occurrences of “HHH” in this word (overlaps are allowed). For example, the word THHHHTTHHHHTH has 3 such occurrences.

Hint: Use linearity of expectation.

Bonus. In a city, there are on average 2.3 children in a family. A randomly chosen child has on average 1.6 siblings. Determine the variance of the number of children in a randomly chosen family.

Extra practice problems (do not hand in):  
Chapter 3: 16,17,18,19,23,26,27

- Let $X$ be a $\text{Bin}(n, p)$ random variable. Show that $\text{Var}(X) = np(1−p)$.
- Let $Y$ be a $\text{Geom}(p)$ random variable. Show that $\text{Var}(Y) = (1−p)/p^2$.

Hint: First compute $E[X(X − 1)]$, and then use properties of expectation.