1. Let $X \sim \mathcal{N}(2, 4)$ be a normal random variable. Using only the values of the c.d.f. of a standard normal random variable (https://en.wikipedia.org/wiki/Standard_normal_table#Cumulative), compute the following probabilities:

(a) $P(X < 6)$.
(b) $P(X \leq 6)$.
(c) $P(X < 4|X > 3)$.
(d) $P(X \leq 1)$.
(e) $P(X < -1|X > 1)$.

**Solution:** Let $\Phi(z)$ be the c.d.f. of the standard normal RV. Then the c.d.f. of $X$ is $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \Phi(\frac{1}{2}x - 1)$

(a) $P(X < 6) = \Phi(\frac{6-2}{2}) = 0.97725$
(b) $P(X \leq 6) = P(X < 6) = 0.97725$, since $X$ is a continuous RV.
(c) $P(X < 4|X > 3) = \frac{P(X \in [3, 4])}{P(X > 3)} = \frac{P(X-2 \in [3/2, 4/2])}{1 - P(X \leq 3)} = \frac{\Phi(1) - \Phi(\frac{1}{2})}{1 - \Phi(\frac{1}{2})} = 0.4857$
(d) $P(X \leq 1) = \Phi(-\frac{1}{2}) = 1 - \Phi(+\frac{1}{2}) = 0.30854$
(e) $P(X < -1|X > 1) = \frac{P(X-2 \in [-\frac{3}{2}, -\frac{1}{2}])}{1 - P(X \leq 1)} = \frac{\Phi(-\frac{3}{2}) - \Phi(-\frac{1}{2})}{1 - \Phi(-\frac{1}{2})} = \frac{\Phi(\frac{3}{2}) - \Phi(\frac{1}{2})}{\Phi(\frac{1}{2})} = 0.35$

2. Let $X \sim \mathcal{N}(-1, 1)$.
(a) Find $c$ such that $P(X > c) = \frac{1}{4}$.
(b) Compute $E(X^2)$.

**Solution:** (a) We have to solve

$$P(X > c) = 1 - \Phi(c + 1) = \frac{1}{3},$$
or, in other words, $\Phi(c + 1) = \frac{2}{3}$. Looking at the table, we see that $\Phi(0.43) = 0.66640$, so $c + 1 \approx 0.43$, and $c \approx -0.57$.
(b) Since $\sigma^2(X) = E(X^2) - (E(X))^2$, we have

$$E(X^2) = \sigma^2 + \mu^2 = 1 + (-1)^2 = 2.$$
3. Let $\mu \in \mathbb{R}$ and $\sigma > 0$ and let $X \sim N(\mu, \sigma^2)$. Let $a, b \in \mathbb{R}$ and define the random variable $Y = aX + b$. Show that $Y \sim N(a\mu + b, a^2\sigma^2)$.

**Solution:** Let $X \sim N(\mu, \sigma^2)$ and let $Y = aX + b$. Then $Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$. The cumulative distribution function of $Y$ is

$$F_Y(x) = \mathbb{P}(Y \leq x) = \mathbb{P}(aX + b \leq x)$$

$$= \mathbb{P}(Z \leq \frac{x - (a\mu + b)}{a\sigma})$$

$$= \Phi \left( \frac{x - (a\mu + b)}{a\sigma} \right).$$

Thus

$$f_Y(x) = F_Y'(x) = \frac{1}{a\sigma} \varphi \left( \frac{x - (a\mu + b)}{a\sigma} \right),$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

is the p.d.f. of a standard normal random variable. We obtained that $f_Y(x)$ is the density function of a normal random variable with mean $a\mu + b$ and variance $(a\sigma)^2$, so $Y \sim N(a\mu + b, (a\sigma)^2)$.

4. Suppose $X, Y$ are two discrete RV’s with joint p.m.f. given by the table:

<table>
<thead>
<tr>
<th>$X \downarrow Y \rightarrow$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1/12</td>
<td>1/8</td>
<td>1/8</td>
<td>1/12</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/12</td>
<td>1/9</td>
<td>1/9</td>
</tr>
<tr>
<td>6</td>
<td>1/12</td>
<td>1/12</td>
<td>0</td>
<td>1/9</td>
</tr>
</tbody>
</table>

(a) Calculate the marginal p.m.f. of $X$ and of $Y$.
(b) Calculate $\mathbb{P}(X^2 + Y < 3)$.
(c) Are $X$ and $Y$ independent?

**Solution:**
(a) We have

$$p_X(1/2) = 5/12, \quad p_X(1) = 11/36, \quad p_X(6) = 5/18,$$

and

$$p_Y(0) = 1/6, \quad p_Y(1) = 7/24, \quad p_Y(2) = 17/72, \quad p_Y(3) = 11/36.$$

(b) We have

$$\mathbb{P}(X^2 + Y < 3) = \mathbb{P}((X, Y) \in \{(1/2, 0), (1/2, 1), (1/2, 2), (1, 0), (1, 1)\})$$

$$= \frac{1}{12} + \frac{1}{8} + \frac{1}{8} + \frac{1}{12} = \frac{5}{12}.$$
(c) Since the p.m.f’s are never zero, if the RV’s were independent, the joint p.m.f.
would never be zero either. But since there are two zeros in the table, this is not
the case. Thus, the RV’s are dependent.

5. You have two dice, one with three sides labeled 0, 1, 2 and one with 4 sides,
labeled 0, 1, 2, 3. Let $X_1$ be the outcome of rolling the first die, and $X_2$ the outcome
of rolling the second. The rolls are independent.
(a) What is the joint distribution of $(X_1, X_2)$
(b) Let $Y_1 = X_1 \cdot X_2$ and $Y_2 = \max\{X_1, X_2\}$. Make a table for the joint distribution
function of $(Y_1, Y_2)$.
(c) Compute the marginal distributions of $Y_1, Y_2$. Are $Y_1, Y_2$ independent?

Solution:
(a) By independence we have
$$p(x, y) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$
for all $x \in \{0, 1, 2\}$ and $y \in \{0, 1, 2, 3\}$.


\begin{table}
<table>
<thead>
<tr>
<th>$Y_1 \downarrow Y_2$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$p_{Y_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/12</td>
<td>1/6</td>
<td>1/6</td>
<td>1/12</td>
<td>1/2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/12</td>
<td>0</td>
<td>0</td>
<td>1/12</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>1/6</td>
<td>0</td>
<td>1/6</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/12</td>
<td>1/12</td>
</tr>
<tr>
<td>4</td>
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<td>0</td>
<td>1/12</td>
<td>0</td>
<td>1/12</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/12</td>
<td>1/12</td>
</tr>
<tr>
<td>$p_{Y_2}$</td>
<td>1/12</td>
<td>1/4</td>
<td>5/12</td>
<td>1/4</td>
<td></td>
</tr>
</tbody>
</table>
\end{table}

(c) For the marginal distributions see the margins of the above table. Since
$$P(Y_1 = 1, Y_2 = 0) = 0 \neq P(Y_1 = 1)P(Y_2 = 0),$$
the variables $Y_1$ and $Y_2$ are not independent.

6. The random variables $X, Y$ have joint probability density function
$$f(x, y) = \begin{cases} 
Cy e^{-y-x/y} & \text{if } x > 0 \text{ and } y > 0, \\
0 & \text{otherwise.}
\end{cases}$$
(a) What is the value of $C$? Hint: Integrate with respect to $x$ first.
(b) Find the marginal probability density function $f_Y$.
(c) Compute $P(X \leq Y^2)$.
(d)* Compute $P(X \leq Y^3)$
Solution:
(a) Using that \( \int_0^\infty y^2 e^{-y} dy = 2 \) we obtain that
\[
1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = C \int_0^\infty \int_0^\infty ye^{-y-x/y} dx dy = C \int_0^\infty y^2 e^{-y} dy = 2C,
\]
thus \( C = 1/2. \)

(b) For \( y > 0 \) we have
\[
f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \frac{1}{2} ye^{-y} \int_{0}^{\infty} e^{-x/y} dx = \frac{1}{2} y^2 e^{-y},
\]
and \( f_Y(y) = 0 \) if \( y \leq 0. \)

(c) We have
\[
P(X \leq Y^2) = \frac{1}{2} \int_0^\infty \int_0^{y^2} ye^{-y-x/y} dx dy
= \frac{1}{2} \int_0^\infty y^2 e^{-y}(1-e^{-y}) dy = \frac{7}{8}
\]

(d) We have
\[
P(X \leq Y^3) = \frac{1}{2} \int_0^\infty \int_0^{y^3} ye^{-y-x/y} dx dy
= \frac{1}{2} \int_0^\infty y^2 e^{-y} - y^2 e^{-y-y^2} dy
= 1 - \frac{1}{2} \int_0^\infty y^2 e^{-y-y^2} dy
= 1 - \frac{1}{2} e^{1/4} \int_0^\infty \left[ y(y + \frac{1}{2}) - \frac{1}{2}(y + \frac{1}{2}) + \frac{1}{4}e^{-(y+\frac{1}{2})^2} \right] dy
= 1 - \frac{1}{2} e^{1/4} \left[ -\frac{1}{2} ye^{-(y+\frac{1}{2})^2} + \frac{1}{4} e^{-(y+\frac{1}{2})^2} \right]_0^\infty - \frac{3}{8} e^{1/4} \int_0^\infty e^{-(y+\frac{1}{2})^2} dy
= \frac{9}{8} - \frac{3}{8} e^{1/4} \int_0^\infty e^{-(y+\frac{1}{2})^2} dy
= \frac{9}{8} - \frac{3}{8} e^{1/4} \sqrt{\pi} P(X > 0),
\]
where \( X \sim \mathcal{N}(-1/2, 1/2). \) Since \( P(X > 0) = 1 - \Phi(1/\sqrt{2}) \), we get
\[
P(X \leq Y^3) = \frac{9}{8} - \frac{3}{8} e^{1/4} (1 - \Phi(1/\sqrt{2})) = 0.92
\]

7. Let \( X \sim \text{Exp}(1/2), Y \sim \text{Unif}([2, 4]) \), and assume that \( X \) and \( Y \) are independent. Calculate \( P(Y - X \geq \frac{1}{2}) \).
Solution: The joint density function is
\[ f(x, y) = f_X(x)f_Y(y) = \begin{cases} \frac{1}{4}e^{-\frac{1}{2}x} & \text{if } x > 0 \text{ and } 2 < y < 4, \\ 0 & \text{otherwise.} \end{cases} \]

Let \( T \) be defined by
\[ T = \{(x, y) : x > 0, \ 2 < y < 4, \ x \leq y - \frac{1}{2} \}, \]
then we have
\[ P(Y - X \geq 1/2) = \int_T f(x, y) \, dy \, dx \]
\[ = \int_2^4 \int_0^{y - \frac{1}{2}} \frac{1}{4}e^{-\frac{1}{2}x} \, dx \, dy \]
\[ = \frac{1}{2} \int_2^4 1 - e^{-y/2+1/4} \, dy \]
\[ = 1 + e^4 \left[ e^{-y/2} \right]_2^4 = 1 + e^{-7/4} - e^{-3/4}. \]

8. Let \( Z_1 \) and \( Z_2 \) be two points chosen uniformly from the unit disk, independently of each other. Let \( d(Z_1, Z_2) \) denote their Euclidean distance, that is, if \( z_i = (x_i, y_i) \), then \( d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \). Compute \( E(d(Z_1, Z_2)^2) \).

Solution: Since the points are chosen uniformly, the p.d.f. of \( Z_i, i = 1, 2 \) is
\[ f_i(x, y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases} \]
By independence, the joint p.d.f. of \((Z_1, Z_2)\) is
\[ f((x_1, y_1), (x_2, y_2)) = \begin{cases} \frac{1}{\pi^2} & x_1^2 + y_1^2 \leq 1 \text{ and } x_2^2 + y_2^2 \leq 1 \\ 0 & \text{otherwise} \end{cases} \]
We compute
\[ E(d(Z_1, Z_2)^2) = \frac{1}{\pi^2} \int_{x_1^2+y_1^2 \leq 1} \int_{x_2^2+y_2^2 \leq 1} [(x_1 - x_2)^2 + (y_1 - y_2)^2] \, dx_1 \, dy_1 \, dx_2 \, dy_2 \]
\[ = \frac{1}{\pi^2} \int_{x_1^2+y_1^2 \leq 1} \int_{x_2^2+y_2^2 \leq 1} [x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2x_1x_2 - 2y_1y_2] \, dx_1 \, dy_1 \, dx_2 \, dy_2 \]
\[ = \frac{4}{\pi^2} \int_{x_1^2+y_1^2 \leq 1} \int_{x_2^2+y_2^2 \leq 1} x_1^2 \, dx_1 \, dy_1 \, dx_2 \, dy_2, \]
where we have used symmetry. We calculate

\[
\int \int_{x^2_1 + y^2_1 \leq 1} \int \int_{x^2_2 + y^2_2 \leq 1} x_1^2 \, dx_1 \, dy_1 \, dx_2 \, dy_2 = \pi \int \int_{x^2_1 + y^2_1 \leq 1} x_1^2 \, dx_1 \, dy_1
\]

\[
= \pi \int_{-1}^{1} x_1^2 \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} dy_1 \, dx_1
\]

\[
= \pi \int_{-1}^{1} 2x_1^2 \sqrt{1-x_1^2} \, dx_1 = \frac{\pi^2}{4}
\]

and conclude \( E(d(Z_1, Z_2)^2) = 1 \).

Second solution: Denote \( E(d(Z_1, Z_2)^2) = E(X_1^2 + X_2^2 + Y_1^2 + Y_2^2 - 2X_1X_2 - 2Y_1Y_2) = 4E(X_1^2) \), by symmetry, and

\[
E(X_1^2) = \int x_1^2 f_{X_1}(x_1) \, dx_1 = \frac{1}{\pi} \int_{-1}^{1} 2x_1^2 \sqrt{1-x_1^2} \, dx_1 = \frac{1}{4}
\]