1. Suppose that the continuous RV $X$ has c.d.f. given by

$$F(x) = \begin{cases} 
0 & \text{if } x < \frac{1}{\sqrt{2}} \\
5 - 12\sqrt{2}x + 18x^2 - 4\sqrt{2}x^3 & \text{if } \frac{1}{\sqrt{2}} \leq x < \sqrt{2} \\
1 & \text{if } \sqrt{2} \leq x
\end{cases}$$

(a) Find the smallest interval $[a, b]$ such that $P(a \leq X \leq b) = 1$.

(b) Find $P(0 < X < \frac{1}{2})$.

(c) Find $P(X = 1)$.

(d) Find $P(1 \leq X \leq \frac{3}{2})$.

(e) Find the p.d.f. of $X$.

Solution: a) Since $P([a, b]) = F(b) - F(a)$ the smallest such interval is $[\frac{1}{\sqrt{2}}, \sqrt{2}]$.

b) Since $\frac{1}{2} \leq \frac{1}{\sqrt{2}}$, $P(0 < X < \frac{1}{2}) = 0$.

c) Since $X$ is a continuous RV,

$$P(X = 1) = 0.$$ 

Remark: The fact that $X$ is indeed continuous can be read off from the fact that $F$ is continuous.

d) 

$$P(1 \leq X \leq 3/2) = F(3/2) - F(1) = \frac{45}{2} - \frac{27}{\sqrt{2}} - 2\sqrt{2}$$

e) 

$$f(x) = F'(x) = \begin{cases} 
-12\sqrt{2} + 36x - 12\sqrt{2}x^2 & \text{if } \frac{1}{\sqrt{2}} \leq x < \sqrt{2}, \\
0 & \text{otherwise.}
\end{cases}$$

Note that actually $F(x)$ is actually continuously differentiable at $x = \frac{1}{\sqrt{2}}$ and $\sqrt{2}$, but in principle we can define $f(x)$ arbitrarily there, it won’t change the integrals.

2. Define the function

$$f(x) = \begin{cases} 
9x^2 - 4x^3 + b & x \in [0, 1] \\
0 & \text{otherwise}
\end{cases}$$

Show that there is no value of $b$ for which this is the p.d.f. of some continuous RV.

Solution: First $f(0) \geq 0$, so $b \geq 0$. We also need $\int_{-\infty}^{\infty} f(x) \, dx = 1$, so

$$1 = \int_0^1 9x^2 - 4x^3 + b \, dx = 2 + b.$$ 

Thus we have $b = -1$ which does not satisfy $b \geq 0$, so $f$ is not a p.d.f. for any $b$. 

3. Suppose a continuous RV $X$ has the p.d.f.

$$f(x) = \begin{cases} \frac{c}{1+x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(a) Find the c.d.f. of $X$.
(b) What must be the value of $c$?
(c) Find $E[X]$.
(d) Compute $E\left(\frac{1}{\sqrt{1 + X^2}}\right)$.

**Solution:**
(a) The c.d.f. is the integral of the p.d.f., and so, according to integration tables, $F(x) = c \arctan x + C$ if $x > 0$ and 0 otherwise. The constant of integration has to be $C = 0$, for $F$ to be continuous at 0.

(b) We need $F(\infty) = 1$, and this means that $c^{-1} = \arctan \infty = \frac{\pi}{2}$, or $c = \frac{2}{\pi}$.

(c) By definition of the expectation, we have

$$E[X] = \int_{0}^{\infty} x \cdot \frac{2}{\pi} \frac{1}{1 + x^2} \, dx$$

This integral is divergent, and therefore $X$ does not have a well defined average (it is too spread out). We will learn later in the course how this is to be interpreted.

(d)

$$E\left(\frac{1}{\sqrt{1 + X^2}}\right) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{1 + x^2}} f(x) \, dx = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{(1 + x^2)^{3/2}} \, dx = \frac{2}{\pi} \left[ \frac{x}{\sqrt{1 + x^2}} \right]_{0}^{\infty} = \frac{2}{\pi}.$$ 

4. Let $X$ be a random variable with p.d.f.

$$f(x) = \begin{cases} 2x^{-2} & x > 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the c.d.f. of $X$.
(b) Find $P(X > 3)$.
(c) Find $P(X > 3 | X < 5)$.
(d) Calculate $E\sqrt{X}$

**Solution:**
(a)

$$F(b) = \int_{-\infty}^{b} f(x) \, dx = \begin{cases} 0 & b < 2 \\ \int_{2}^{b} 2x^{-2} \, dx & b \geq 2 \end{cases} = \begin{cases} 0 & b < 2 \\ \left[ 2b^{-1} \right]_{2}^{b} & b \geq 2 \end{cases} = \begin{cases} 0 & b < 2 \\ 1 - 2b^{-1} & b \geq 2 \end{cases}$$

(b)

$$P(X > 3) = 1 - F(3) = \frac{2}{3}$$

(c)

$$P(X > 3 | X < 5) = \frac{P(\{X > 3\} \cap \{X < 5\})}{P(X < 5)} = \frac{P(X \in (3,5))}{P(X < 5)} = \frac{F(5) - F(3)}{F(5)} = \frac{4}{9}$$
6. Let $c > 0$ and $X \sim \text{Unif}[0, c]$. Show that the RV $Y = c - X$ has the same c.d.f. and therefore also the same p.d.f. as $X$.

**Solution:** We have

$$
P(Y \leq b) = P(X \geq c - b) = \begin{cases} 
0 & b \leq 0 \\
\int_{c-b}^{c} \frac{1}{c} \ dt = \frac{b}{c} & 0 \leq b \leq c \\
1 & c \leq b
\end{cases},
$$

which is the same as $P(X \leq b)$. Since the p.d.f. is the derivative of the cdf, also the p.d.f.'s of $X$ and $Y$ coincide.

7. A stick of length $\ell$ is broken into two pieces at a position $X \sim \text{Unif}[0, \ell]$. Let $Y$ denote the length of the smaller piece.

(a) Calculate the c.d.f. of $Y$, that is, calculate $P(Y \leq b)$.

(b) Calculate the p.d.f. of $Y$. Can you identify what kind of random variable $Y$ is?

**Solution:**

(a) The length of the smaller piece takes values $Y \in [0, \ell/2]$. By geometric considerations, the c.d.f. is

$$
F_Y(y) = P(Y \leq y) = P(X \leq y \ or \ X \geq \ell - y) = P(X \leq y) + P(X \geq \ell - y) = \int_0^y \frac{1}{\ell} \ dt + \int_{\ell-y}^{\ell} \frac{1}{\ell} \ dt = \frac{2y}{\ell}.
$$

Thus

$$
F_Y(y) = \begin{cases} 
0 & \text{if } y < 0, \\
\frac{2y}{\ell} & \text{if } 0 \leq y \leq \ell/2, \\
1 & \text{if } y > \ell/2.
\end{cases}
$$

(b) We have

$$
f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 
\frac{2}{\ell} & \text{if } 0 \leq y \leq \ell/2, \\
0 & \text{otherwise}.
\end{cases}
$$

We recognize that $Y \sim \text{Unif}[0, \ell/2]$

8. Let $X$ be an Exp(4) random variable. Find a number $a$ such that $\{X \in [0, 1]\}$ is independent of $\{X \in [a, 2]\}$
Solution: If $a < 0$ then all probabilities are the same as in the case $a = 0$, so we may assume that $0 \leq a \leq 1$. We have
\[
P(X \in [0, 1]) = F_X(1) - F_X(0) = 1 - e^{-4},
\]
and
\[
P(X \in [a, 2]) = F_X(2) - F_X(a) = e^{-4a} - e^{-8}.
\]
The probability of intersection is
\[
P(X \in [0, 1], X \in [a, 2]) = P(X \in [a, 1]) = F_X(1) - F_X(a) = e^{-4a} - e^{-4}.
\]
The definition of independence gives the equation
\[
e^{-4a} - e^{-4} = (1 - e^{-4})(e^{-4a} - e^{-4}),
\]
so
\[
e^{-4a} = 1 - e^{-4}(1 - e^{-4}),
\]
that is,
\[
a = -\frac{1}{4} \ln(1 - e^{-4}(1 - e^{-4})) \approx 0.0045.
\]

9. (a) Suppose that the duration $T$ (in hours) of your morning routine (breakfast, shower, etc.) is modeled by an exponential RV with parameter $\lambda$. You set your alarm 1 hour before your bus leaves for UBC. For which value of $\lambda$ do you have a 50% chance of catching the bus?

(b)* Calculate the $n$th moment of $T$.

Solution: a) We are interested for which $\lambda$ we have $P(T \leq 1) = \frac{1}{2}$. We compute
\[
P(T \leq 1) = \int_0^1 \lambda e^{-\lambda x} \, dx = 1 - e^{-\lambda},
\]
and so $\lambda = \ln 2$.

b) We have
\[
E X^n = \int_0^\infty x^n \cdot \lambda e^{-\lambda x} \, dx = \lambda^{-n} \int_0^\infty x^n e^{-x} \, dx
\]
Let us call the latter integral $I_n$. Using integration by parts, we have
\[
I_n = \left[ -x^n e^{-x} \right]_0^\infty + n I_{n-1} = n I_{n-1}
\]
if $n \geq 1$. Since $I_0 = 1$ by the normalization property (or computation), we conclude that $I_n = n!$, and $E X^n = \lambda^{-n} n!$. 