Math 302, Practice Assignment

1. Let $X$ be a geometric random variable with parameter $\frac{1}{5}$.
   (a) Use Chebyshev’s inequality to find an upper bound for $\Pr(X \geq 15)$.
   (b) Compute $\Pr(X \geq 15)$ exactly and compare with the result in (a).

   **Solution:**
   a) As $\mathbb{E}(X) = 5$ and $\text{Var}(X) = 20$, we have
   $$\Pr(X \geq 15) \leq \Pr(|X - \mathbb{E}(X)| \geq 10) \leq \frac{20}{10^2} = 0.2.$$
   b) We have $\Pr(Y \geq 15) = \Pr(Y > 14) = (1 - 1/5)^{14} \approx 0.04.$

2. In a lottery, you can win a nice smartphone ($1200). Each ticket costs $20, and
   4% of tickets are a win. You decide to buy tickets until you win the phone.
   (a) Give a lower bound on the probability that this is better than just buying the
   phone in store.
   (b) Give a price that your costs can exceed with probability at most 25%.

   **Solution:**
   (a) The number of tickets you have to buy is a geometric RV $X$ with parameter
   $p = 0.04$. It’s expectation and variance are $\mathbb{E}X = 25$ and $\sigma^2(X) = 600$. The event
   that it is better to play the lottery than buying the phone in store is
   $\{20X \leq 1200\} = \{X \leq 60\}$. By Chebyshev’s inequality,
   $$\Pr(X \leq 60) \geq 1 - \frac{600}{35^2} = \frac{25}{49} \approx 51%$$
   So in at least half of the cases, it is better to play this lottery than to buy the phone
   in store.

   (b) We need to find a number $P$ such that $\Pr(20X \geq P) \leq 0.25$. Using Chebyshev,
   we know that
   $$\Pr(20X \geq P) = \Pr(X - 25 \geq \frac{P}{20} - 25) \leq \Pr(|X - 25| \geq \frac{P}{20} - 25) \leq 0.25$$
   if $\varepsilon = \frac{P}{20} - 25$ satisfies $\frac{600}{35^2} = 0.25$. We conclude that $\varepsilon = \sqrt{2400}$, or $P =
   20(\sqrt{2400} + 25) \approx 1479.80.$

3. Copying 1 billion ($10^9$) bits from a USB drive to your hard disk takes about
   0.2 seconds under ideal conditions. However, every bit has a probability of roughly
   $p = 10^{-8}$ to be copied incorrectly. Using error correcting codes, your PC is able
   to recognize and correct flawed bits during the copying process. Assume that it
   takes 0.02 seconds to correct a flawed bit. We are interested in the probability that
   copying a movie of 2.5 gigabytes (a byte is 8 bits, and giga is a billion) takes less
   than 9 seconds.
   (a) Write down a precise expression for this probability.
   (b) Give a lower bound on this probability.
   For the next two questions, use the Poisson approximation for the number of flawed
   bits.
(c) Give a lower bound on this probability using Chebyshev’s inequality.
(d) Compute this probability.

**Solution:** Let $X$ denote the number of flawed bits when copying the movie. Since 2.5 gigabytes are $2 \cdot 10^{10}$ bits, we have that $X \sim Bin(2 \cdot 10^{10}, 10^{-8})$ and the time consumed by the raw copying plus the error correction is $T = 4 + 0.02 \cdot X$ seconds.

(a) We have

$$
\mathbb{P}(T \leq 9) = \mathbb{P}(X \leq 250) = \sum_{k=0}^{250} \binom{2 \cdot 10^{10}}{k} \cdot (10^{-8})^k \cdot (1 - 10^{-8})^{2 \cdot 10^{10} - k}.
$$

This expression involves rather large numbers – even a computer will have some difficulty computing its value.

(b) We have

$$
\mathbb{P}(T \leq 9) = \mathbb{P}(X \leq 250) = 1 - \mathbb{P}(X > 250) \geq 1 - \mathbb{P}(|X - \mathbb{E}X| > 50).
$$

Since $\mathbb{E}X = 2 \cdot 10^{10} \cdot 10^{-8} = 200$ and $\text{Var}(X) = 200 \cdot (1 - 10^{-8})$, Chebyshev’s inequality yields that

$$
\mathbb{P}(|X - \mathbb{E}X| > 50) \leq \frac{\text{Var}(X)}{50^2} = \frac{2}{25} \cdot (1 - 10^{-8}) \approx 8\%.
$$

Thus, $\mathbb{P}(T \leq 9)$ is at least 92%.

(c) We now approximate $X$ by a r.v. $X' \sim \text{Poisson}(200)$. Then, similarly to before, since $\mathbb{E}X' = 200$ and $\text{Var}(X') = 200$, Chebyshev’s inequality yields that

$$
\mathbb{P}(|X' - \mathbb{E}X'| > 50) \leq \frac{\text{Var}(X')}{50^2} = \frac{2}{25} = 8\%.
$$

Thus, we recover the same lower bound as in (b).

(d) We have

$$
\mathbb{P}(X' \leq 250) = \sum_{k=0}^{250} \frac{200^k}{k!} e^{-200}.
$$

We can use the computer to evaluate this number and get approximately 99.97%. This yields that $\mathbb{P}(T \leq 9)$ is approximately 99.97%.

4. Let $X$ be a Poisson random variable with parameter $\lambda$

a) Which $n = n(\lambda) \geq 0$ is the most likely value of $X$, i.e. maximizes $\mathbb{P}(X = n)$?

b) Suppose the experiment described by $X$ has returned the value $n \geq 0$. Which parameter $\lambda = \lambda(n)$ maximizes $\mathbb{P}(X = n)$?

**Solution:** a) Let $g(n) = \mathbb{P}(X = n) = e^{-\lambda} \frac{\lambda^n}{n!}$. Then we have

$$
g(n) = \frac{\lambda}{n} g(n-1),
$$

so $g(n) > g(n-1)$ if and only if $n < \lambda$ and $g(n) = g(n-1)$ if $n = \lambda$ is integer. Therefore the probability $g(n)$ is maximal if $n \leq \lambda \leq n+1$. (If $\lambda$ is an integer then we have two solutions $n = \lambda$ and $n = \lambda - 1$, otherwise $n = \lfloor \lambda \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer not greater than $x$.)
b) If the function \( f(\lambda) = P(X = n) = e^{-\lambda \frac{\lambda^n}{n!}} \) is maximal at a point \( \lambda > 0 \), then its derivative must be zero at that point, that is,

\[
f'(\lambda) = e^{-\lambda} \frac{n \lambda^{n-1} - \lambda^n}{n!} = 0.
\]

This gives that \( \lambda = n \). Clearly \( f'(x) > 0 \) if \( \lambda < n \) and \( f'(x) < 0 \) if \( \lambda > n \), so \( f \) is increasing on the interval \([0, n]\) and decreasing on \([n, \infty)\). Thus there is really a global maximum at \( \lambda = n \).