1. Let $X$ take values in $\{1, 2, 3, 4, 5\}$ and have p.m.f. given by

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = k)$</td>
<td>$1/7$</td>
<td>$1/14$</td>
<td>$3/14$</td>
<td>$2/7$</td>
<td>$2/7$</td>
</tr>
</tbody>
</table>

(a) Calculate $P(X \leq 3)$
(b) Calculate $P(X < 3)$
(c) Calculate $P(X < 4.12 \mid X > 1.6)$
(d) Calculate $E(X)$
(e) Calculate $E[X - 2]$
(f) Calculate $\text{Var}(X)$

2. An urn contains 5 black and 5 white balls. You draw balls one by one, until you draw your first white ball. The total number of balls you have drawn, including the first white ball, is a random variable $X$.

(a) Calculate the p.m.f. of $X$.
(b) Calculate the expectation $E(X)$.
(c) Calculate the variance $\text{Var}(X)$.
(d) Suppose that a white ball is worth $4$, but that you have to double your investment for every ball you draw, starting at $1$ to draw the first ball (i.e., you have to pay $2$ to draw two balls, $4$ to draw 3 balls, and so on). What are your expected gains or losses in this game?

3. Prove the following claims from the lecture. Here, $X, Y$ are discrete random variables on the same sample space, and $a, b \in \mathbb{R}$.

(a) $E(ax + b) = aE(X) + b$
(b) $\text{Var}(ax + b) = a^2\text{Var}(X)$
(c) $\text{Var}(X) = E(X^2) - (E(X))^2$
(d) $E(X + Y) = E(X) + E(Y)$

4. Let $X$ be a $\text{Bin}(n, p)$ random variable. Show that $\text{Var}(X) = np(1 - p)$.

Hint: First compute $E[X(X - 1)]$ and then use (c) and (d) from question 3.

5. Let $X$ be a discrete random variable. Define a function $f: \mathbb{R} \to \mathbb{R}$ by

$$f(a) = E[(X - a)^2].$$

Show that $f$ attains its minimum at $a = E(X)$. What is $f(E(X))$? Try to explain what this means.

6. A fair coin is tossed 50 times. The outcomes are written in order, producing a 50-letter “word” consisting of the letters H and T. Compute the expected number of occurrences of HHH in this word (overlaps are allowed).

For example, the word THHHHTTHHHTH has 3 such occurrences.

7*. In a town, there are on average 2.3 children in a family and a randomly chosen child has on average 1.6 siblings. Determine the variance of the number of children in a randomly chosen family.