1. Consider the following game: An urn contains 20 white balls and 10 black balls. If you draw a white ball, you get $1, but if you draw a black ball, you loose $2.
(a) You draw 6 balls out of the urn. What is the probability that you will win money?
(b) How many balls should you draw in order to maximize the probability of winning? Hint: Use a computer.

Solution:
(a) The total number of drawings is \( \binom{30}{6} \). You will win money if you draw either 5 or 6 white balls. Therefore,
\[
P(\text{win money}) = \frac{\binom{20}{5} \cdot \binom{10}{1}}{\binom{30}{6}} + \frac{\binom{20}{6} \cdot \binom{10}{0}}{\binom{30}{6}} = \frac{2584}{7917} \approx 0.33.
\]
(b) If \( n \) is the number of balls you decide to draw, the condition on the number of white balls that need to be present in order for you to make money is \( k - 2(n - k) > 0 \), or \( k > \frac{2}{3}n \). Therefore,
\[
P(\text{win money with } n \text{ draws}) = \sum_{k>\frac{2}{3}n}^{n} \frac{\binom{20}{k} \cdot \binom{10}{n-k}}{\binom{30}{6}}.
\]
Evaluating these values numerically for all possible \( n = 0, \ldots, 30 \), we see that the largest value is achieved at \( n = 1 \), i.e. only choosing a single ball, where the probability of winning is \( \frac{2}{3} \).

2. In a group of 5 teenagers, what is the probability that at least two of them were born in the same year?

Solution: There are 7 years that a group of people that are simultaneously teenagers could have been born in. As in the lecture, the probability that none of them were born in the same year is \( \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7^5} = \frac{360}{2401} \), so the probability in question is \( P = 1 - \frac{360}{2401} = \frac{2041}{2401} \approx 0.85 \).

3. Assume that the events \( E_1, E_2 \) are independent.
   a) Prove that the events \( E_1^c, E_2^c \) are also independent.
   b) If, in addition, \( P(E_1) = \frac{1}{2} \) and \( P(E_2) = \frac{1}{4} \). Compute \( P(E_1 \cup E_2) \).
   c) If, in addition, \( E_3 \) is a third event that is independent of \( E_1 \) and of \( E_2 \), and such that \( P(E_3) = \frac{1}{4} \). Prove that
\[
\frac{17}{24} \leq P(A \cup B \cup C) \leq \frac{19}{24}.
\]
Solution:
(a) We have
\[
P(E_1^c \cap E_2^c) = 1 - P(E_1 \cup E_2)
= 1 - \left( P(E_1) + P(E_2) - P(E_1 \cap E_2) \right)
= 1 - \left( P(E_1) + P(E_2) - P(E_1)P(E_2) \right)
= (1 - P(E_1))(1 - P(E_2)) = P(E_1^c)P(E_2^c)
\]
which proves independence.

(b) There are two ways to do the computation. We may either use inclusion-exclusion according to
\[
P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)
= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3},
\]
or we may use that, by (a), \( E_1^c, E_2^c \) are also independent, which allows us to compute
\[
P(E_1 \cup E_2) = 1 - P(E_1^c \cap E_2^c) = 1 - P(E_1^c)P(E_2^c) = 1 - \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{3}.
\]

(c) Using inclusion/exclusion, we compute
\[
P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3)
+ P(E_1 \cap E_2 \cap E_3)
= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{12} - \frac{1}{8} + P(E_1 \cap E_2 \cap E_3)
= \frac{17}{24} + P(E_1 \cap E_2 \cap E_3)
\]
We used here that the pairs \((E_1, E_2), (E_1, E_3)\) and \((E_2, E_3)\) are independent. Since we do not assume that the triple \(E_1, E_2, E_3\) is independent, we cannot compute the probability of the \(E_1 \cap E_2 \cap E_3\) exactly, but since \(E_1 \cap E_2 \cap E_3\) is contained in all three of \(E_1 \cap E_2, E_1 \cap E_3\) and \(E_2 \cap E_3\), it must have probability smaller than any of these, i.e. \(0 \leq P(E_1 \cap E_2 \cap E_3) \leq \frac{1}{12}\). This immediately gives the claim.

4. Eight rooks are placed randomly on a chess board. What is the probability that none of the rooks can capture any of the other rooks? (In non-chess terms: Randomly pick 8 unit squares from an 8 \times 8 square grid. What is the probability that no two squares share a row or a column?)

**Hint:** How many choices do you have to place rooks in the first row? After you have made your choice, how many choices do you have for the second? Continue this reasoning.

**Solution:** The total number of choosing 8 positions for the rooks on a board with 64 fields is \(\binom{64}{8}\). The number of favorable outcomes is 8!; there are 8 possibilities two choose a square from the first row, 7 ways to choose one from the second row, and so on. Thus our probability in question is
\[
P = \frac{8!}{\binom{64}{8}} = \frac{(8!)^2}{64 \cdots 57}.
\]

5. We toss two dice. Consider the events
E: The sum of the outcomes is odd.
F: At least one outcome is 4.
Calculate the conditional probabilities \( P(E | F) \) and \( P(F | E) \).

**Answer:** 
\[ P(E) = \frac{1}{2}, \quad P(F) = \frac{11}{36}, \quad P(E \cap F) = \frac{6}{36}, \] so 
\[ P(E | F) = \frac{P(E \cap F)}{P(F)} = \frac{6/11}{11/36} = \frac{6}{11} \] and 
\[ P(F | E) = \frac{P(E \cap F)}{P(E)} = \frac{6}{11}. \]

6. A fair die is rolled repeatedly.
(a) Give an expression for the probability that the first five rolls give a three at most two times.
(b) Calculate the probability that the first three does not appear before the fifth roll.
(c) Calculate the probability that the first three appears before the twentieth roll, but not before the fifth roll.

**Solution:**

a) Let \( X \) denote the number of threes on the first five rolls, then our probability is 
\[ P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \binom{5}{0} \left( \frac{5}{6} \right)^5 + 5 \binom{5}{1} \left( \frac{5}{6} \right)^4 \left( \frac{1}{6} \right) + 10 \binom{5}{2} \left( \frac{5}{6} \right)^3 \left( \frac{1}{6} \right)^2. \]

b) The event in question is that none of the first four rolls is a three. On each die the probability of not rolling three is \( \frac{5}{6} \), so by independence the probability in question is \( \left( \frac{5}{6} \right)^4 \).

c) Let \( A \) be the event that none of the first four rolls is a three, and \( B \) be the event that some of the rolls from 5–19 is a three, then our event in question is \( A \cap B \). By part b) we have \( P(B) = (5/6)^4 \) and similarly we obtain \( P(B^c) = (5/6)^{15} \), so \( P(B) = 1 - (5/6)^{15} \). Since \( A \) and \( B \) are independent, we obtain 
\[ P(A \cap B) = P(A)P(B) = (5/6)^4 \left( 1 - (5/6)^{15} \right). \]

7.* Let the sequence of events \( E_1, E_2, \ldots, E_n \) be independent, and assume that 
\( P(E_i) = \frac{1}{i+1} \). Show that 
\[ P(E_1 \cup \cdots \cup E_n) = \frac{n}{n+1}. \]

**Solution:** The sequence \( E_1^c, E_2^c, \ldots, E_n^c \) is also independent. This was shown for \( n = 2 \) in problem 3 above, and is a special case of Fact 2.23 in the textbook. It can be proven by induction, using the inclusion/exclusion formula. From this fact, the claim follows by the calculation 
\[ P(E_1 \cup \cdots \cup E_n) = 1 - P(E_1^c \cap \cdots \cap E_n^c) \]
\[ = 1 - P(E_1^c) \cdots P(E_n^c) = 1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n}{n+1} \]
\[ = 1 - \frac{1}{n+1} = \frac{n}{n+1}. \]