1. Let $\Omega = \{1, \emptyset, c\}$ be a sample space. List all possible events.

Solution: $\emptyset$, $\{1\}$, $\{\emptyset\}$, $\{1, \emptyset\}$, $\{1, c\}$, $\{\emptyset, c\}$.

2. Your baking cupboard contains 1 cup of whole wheat flour, 1 cup of white flour, 1 cup of brown sugar, 1 cup of white sugar, and 2 eggs. Consider the following random baking experiment: You thoroughly mix three randomly chosen ingredients in a bowl and throw it into the oven.

(a) Write down the sample space of this experiment.

(b) What is the probability that you will actually bake a cake? (A super basic sponge cake, anyway).

Solution:

(a) Denote by $U = \{F_1, F_2, S_1, S_2, E_1, E_2\}$ the collection of flour 1, flour 2, sugar 1, sugar 2, egg 1 and egg 2 in the cupboard. Then $S = \{D \subseteq U, |D| = 3\}$ is the sample space. It has $|S| = \binom{6}{3} = 20$ elements.

(b) A sponge cake consists of the three ingredients flour (any kind), sugar (any kind) and egg. Let $\phi : U \to \{F, S, E\}$ be the map that returns the type of each ingredient (flour, sugar or egg), i.e. $\phi(F_1) = \phi(F_2) = F$, and so on. Then $E = \{\text{bake a cake}\} = \{D \subseteq S, \phi(D) = \{F, S, E\}\}$. More explicitly,

$$E = \left\{ \{F_i, S_j, E_k\}, i, j, k \in \{1, 2\} \right\}.$$ 

Clearly $|E| = 2^3$, so $P(E) = \frac{8}{20} = \frac{2}{5}$.

3. Let $S$ be a sample space and $P$ be a probability. Prove that there can’t exist events $E, F$ that satisfy

$$P(E \setminus F) = \frac{1}{3}, \quad P(E \cap F) = \frac{1}{4}, \quad \text{and} \quad P(E^c \cap F^c) = \frac{1}{2}.$$ 

Solution: The formula $P(A) = 1 - P(A^c)$ yields that

$$P(E \cup F) = 1 - P(E^c \cap F^c) = \frac{1}{2}.$$ 

The third axiom of probability implies that

$$\frac{1}{2} = P(E \cup F) = P(E \setminus F) + P(E \cap F) + P(F \setminus E) = \frac{1}{3} + \frac{1}{4} + P(F \setminus E),$$ 

but $P(F \setminus E) \geq 0$, which gives a contradiction.

4. Let $A, B, C$ be events in a sample space $\Omega$. Prove that

(a) $P(A \cup B) \leq P(A) + P(B)$.

Solution: Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and $P(A \cap B) \geq 0$, the claim follows immediately.

(b) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

First solution:
Applying \( P(E \cup F) = P(E) + P(F) - P(E \cap F) \) first for \( E = A \cup B \) and \( F = C \), then for \( E = A \) and \( F = B \), finally for \( E = A \cap C \) and \( F = B \cap C \) implies that
\[
P(A \cup B \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)
= P(A) + P(B) + P(C) - P(A \cap B) - P((A \cap C) \cup (B \cap C))
= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)
+ P(A \cap B \cap C).
\]

Second solution hint: An argument similar to the proof of \( P(E \cup F) = P(E) + P(F) - P(E \cap F) \) works, too. Consider the 8 events \( A' \cap B' \cap C' \), where \( A' = A \) or \( A' = A^c \), \( B' = B \) or \( B' = B^c \), and \( C' = C \) or \( C' = C^c \). Then each probability in the equation can be expressed by the probabilities \( P(A' \cap B' \cap C') \). Comparing these equations yields (b).

5. Assuming a fair poker deal, what is the probability of a
   (a) royal flush
   (b) straight flush
   (c) flush
   (d) straight
   (e) two pair

Solution: The number of all poker hands is \( \binom{52}{5} = 2,598,960 \) and they are equally likely.

a) In a given suit there is only one royal flush, there are 4 possible suits, so the probability is
\[
\frac{4}{2,598,960} = \frac{1}{649,740}.
\]
b) In a given suit there are 10 sequences of five in a row, since the lowest value can be \( A, 2, 3, 4, 5, 6, 7, 8, 9, 10 \), but the last one is a royal flush. As there are 4 possible suits, the probability is
\[
\frac{4 \cdot 9}{2,598,960} \approx \frac{1}{72,193}.
\]
c) A given suit contains 13 cards, there are \( \binom{13}{5} = 1,287 \) ways to choose 5. There are 10 which form a sequence, so the probability is
\[
\frac{4 \cdot (1287 - 10)}{2,598,960} \approx \frac{1}{509}.
\]
d) There are 10 possibilities to choose the lowest value of the sequence, then for any value 4 different suit can be assigned. That is \( 10 \cdot 4^5 = 10,240 \) possibilities, from which there are 40 straight flushes and royal flushes. Therefore the probability is
\[
\frac{10,240 - 40}{2,598,960} \approx \frac{1}{255}.
\]
e) There are \( \binom{13}{2} = 78 \) possibilities to choose the values of the two pairs, and there is 11 possibility for the value of the last card. We can choose \( \binom{4}{2} = 6 \) possible suits for both pairs and 4 possible suits for the single card, which gives the probability
\[
\frac{78 \cdot 11 \cdot 6^2 \cdot 4}{2,598,960} \approx \frac{1}{21}.
\]
6. How many different anagrams (rearrangements of letters) can be formed from the letters of COMBINATORICS?

**First solution:** The word COMBINATORICS has 13 letters, but 3 of which (C, O, I) are present twice. Consider now instead the string
\[
\text{C}_1 \text{O}_1 \text{MB}_1 \text{N}_1 \text{A}_1 \text{T}_2 \text{O}_2 \text{R}_1 \text{C}_2 \text{S},
\]
where each letter is unique. This string has $13!$ anagrams, and any anagram of COMBINATORICS corresponds to $2^3$ anagrams of
\[
\text{C}_1 \text{O}_1 \text{MB}_1 \text{N}_1 \text{A}_1 \text{T}_2 \text{O}_2 \text{R}_1 \text{C}_2 \text{S},
\]
according to the $2^3$ ways to insert labels $C \rightarrow C_1$, $C \rightarrow C_2$, etc. Therefore, there are $\frac{13!}{2^3}$ anagrams of COMBINATORICS.

**Second solution:** For any of the 13 positions for letters of the anagram, we can choose which of the 10 groups C, O, M, B, I, N, A, T, R, S it should belong to, with the groups corresponding to letters C, O, I of size 2, and all other groups of size 1. Therefore there are
\[
\binom{13}{2,2,2,1,1,1,1,1,1,1} = \frac{13!}{2!2!2!1!1!1!1!1!1!1!} = \frac{13!}{2^3}\text{anagrams of COMBINATORICS}.
\]

7. We toss a fair die four times. What is the probability that all tosses produce different outcomes?

**Solution:** The natural sample space $S$ consists of the (ordered) sequences of 4 tosses, because then every outcome is equally likely. Then $|S| = 6^4 = 1296$. The outcomes with different values correspond the 4-permutations of the 6-element set \{1, \ldots, 6\}, their number is $6 \cdot 5 \cdot 4 \cdot 3 = 360$. (There are 6 options for choosing the first value, 5 for the second and so on.) Thus the probability is $\frac{360}{1296} = \frac{5}{18}$.

8*. Prove that the number of unordered sequences of length $k$ with elements from a set $X$ of size $n$ is $\binom{n+k-1}{k}$.

**Hint:** For illustration, first consider the example $n = 4, k = 6$. Let the 4 elements of the set $X$ be denoted $a, b, c, d$. Argue that any unordered sequence of size 6 consisting of elements $a, b, c, d$ can be represented uniquely by a symbol similar to “···|···|···|···”, corresponding to the sequence $aabccd$. Now count the number of choices for the vertical bars.

**Solution:** Let the elements of $X$ be $\{x_1, \ldots, x_n\}$. We want to count all sequences $(y_1, \ldots, y_k)$ with $y_i \in X$, not counting as different sequences that are obtained from each other via a permutation of the $y_i$’s.

Any sequence $(y_1, \ldots, y_k)$ is therefore equivalent to precisely one where all (if any) $x_1$’s appear first, then all (if any) $x_2$’s and so on. Let’s call a sequence ordered in this way a representative sequence. No two different representative sequences are permutations of each other, and so the number of unordered sequences is exactly equal to the number of representative sequences.

But each representative sequence corresponds uniquely to a symbol of the kind
\[
\cdots \cdots \cdots \cdots, \text{ with exactly } k \text{ dots, and } n-1 \text{ bars.}
\]
Namely, the number of dots before the first bar corresponds to the number of $x_1$’s in the representative sequence, the number of dots between the first and the second bar to the number of $x_2$’s in the sequence, and so on.

Since the symbol \cdots \cdots \cdots \cdots \cdots \cdots \cdots has length number of dots + number of bars $= k + n - 1$, and we may select any of these positions for the $n - 1$ bars. The number of such choices is $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$, proving the claim.