MATH 321:201: Real Variables II (Term 2, 2010)
Home work assignment # 5
Due date: Friday, Feb. 12, 2010 **(hand-in in class)**

Problem 1: Supposed X is a metric space. Let $C(X)$ denote the space of bounded continuous functions $f : X \to \mathbf{R}$. The space $C(X)$ is a metric space with the sup norm metric d_∞:

$$d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Let $\mathcal{F} \subset C(X)$. Suppose the following hold:

(1) X is compact
(2) X is connected
(3) There exists a point $x_0 \in X$ and $M > 0$ such that for all $f \in \mathcal{F}$, $|f(x_0)| \leq M$, i.e. $\mathcal{F}$ is pointwise bounded at x_0.
(4) $\mathcal{F}$ is equicontinuous

Show that $\mathcal{F}$ is uniformly bounded on X. Use this and Arzela-Ascoli theorem, to show $\mathcal{F}$ is pre-compact (with respect to d_∞) in $C(X)$: that is, every sequence in $\mathcal{F}$ has a convergent subsequence (converging to a function in $C(X)$, but not necessarily in $\mathcal{F}$).

Problem 2: Do [Rudin, Ch7. Exercise # 16].

Problem 3: Do [Rudin, Ch7. Exercise # 18].

Problem 4: Do [Rudin, Ch7. Exercise # 20].

The following are suggested exercises. Please DO NOT hand-in, *but, it is important for you to do these suggested exercises!*

Problem: Do Rudin, Ch. 7, Exercises # 15, #17, #19.

Problem: Let $\{f_n\} \subset C(X)$ be a sequence of real-valued continuous functions on a compact domain X. Assume each f_n is Lipschitz and assume that there exists a constant $C > 0$ such that for any f_n, its Lipschitz constant is less than or equal to C, i.e. $|f_n(x) - f_n(y)| \leq Cd(x, y)$, $\forall n \in \mathbf{N}$, $\forall x, y \in X$, where d is the metric of X. Assume further that there is a point $x_0 \in X$ such that the sequence $\{f_n\}$ is pointwise bounded at x_0. Show that there exists a subsequence $\{f_{n_k}\}$ which converges uniformly to a Lipschitz function whose Lipschitz constant is less than or equal to C.