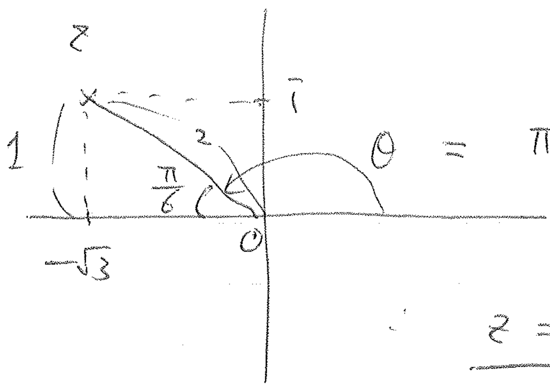
MATH 267. 2011 April

①

Prob 1

$$|z| = \sqrt{3+1} = 2.$$



$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

$$\therefore\ \underline{z = 3e^{i\frac{5\pi}{6}}}.$$

$$z^3 = 3^3 e^{i\frac{5\pi}{6}\times 3} = 27\cdot e^{i\frac{5\pi}{2}}$$

$$= 27\left(e^{i\frac{\pi}{2}}\right)^5$$

$$= 27\, i^5 = 27\, i$$

$$\bar{z}^3 = \overline{z^3} = -27\, i$$

$$\therefore\ z^3 - \bar{z}^3 = 27i - (-27i) = \underline{54\, i}. \quad ☑$$

Prob 2

$$\sum_{k=-15}^{0} e^{-i\frac{2\pi}{15}k} = \sum_{m=0}^{15} e^{-i\frac{2\pi}{15}(m-15)} \qquad m = k+15$$

$$= \sum_{m=0}^{15} e^{-i\frac{2\pi}{15}m} \cdot \underbrace{e^{i\frac{2\pi}{15}15}}_{e^{i2\pi} = 1}.$$

$$= \sum_{m=0}^{15} e^{-i\frac{2\pi}{15}m}$$

$$= \Big[\underbrace{\cancel{\sum_{m=0}^{14} e^{-i\frac{2\pi}{14}m}}}_{0\ (\because\ \text{orthogonality})} + \underbrace{e^{-i\frac{2\pi}{15}15}}_{=1} \Big]$$

$$= 1$$

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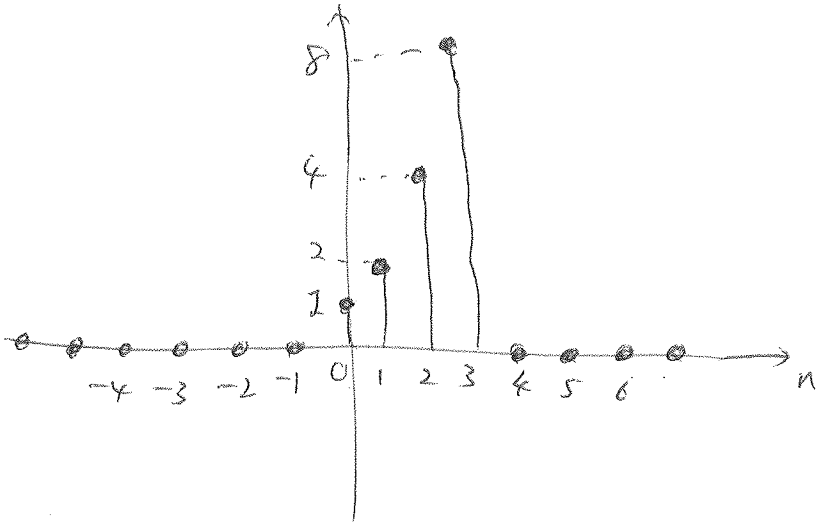
Prob 3



Prob 4

period $= 24\pi$
$= 2L \quad \therefore \quad L = 12\pi$

$$\therefore \ e^{ik\pi t/L} = e^{ik\frac{t}{12}}$$

$$g(t) = \cos\left(\frac{t}{3}\right) + \sin\left(\frac{t}{4}\right)$$

$$= \frac{1}{2}\left(e^{i\frac{t}{3}} + e^{-i\frac{t}{3}}\right) + \frac{1}{2i}\left(e^{i\frac{t}{4}} - e^{-i\frac{t}{4}}\right)$$

$$= \frac{1}{2}\left(e^{i4\frac{t}{12}} + e^{-i4\frac{t}{12}}\right) + \frac{1}{2i}\left(e^{i3\frac{t}{12}} - e^{-i3\frac{t}{12}}\right)$$

$$= \frac{1}{2}e^{-i4\frac{t}{12}} - \frac{1}{2i}e^{-i3\frac{t}{12}} + \frac{1}{2i}e^{i3\frac{t}{12}} + \frac{1}{2}e^{i4\frac{t}{12}}$$

$$\therefore \ \hat{g}(k) = \begin{cases} \frac{1}{2} & k = -4 \\ -\frac{1}{2i} & k = -3 \\ \frac{1}{2i} & k = 3 \\ \frac{1}{2} & k = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Prob 5

Try $X = e^{rx}$;

Then get $r^2 - 2r + (1+\sigma^2) = 0$

$r = 1 + i\sigma, \quad 1 - i\sigma$

$$\therefore X(x) = \cancel{A e^{(1+i\sigma)x} + B e^{(1-i\sigma)x}}$$

$$= A e^{(1+i\sigma)x} + B e^{(1-i\sigma)x}$$

$$= e^{x}\left[A e^{i\sigma x} + B e^{-i\sigma x} \right].$$

- $0 = X(0) = A + B$
- $0 = X(1) = e^{1} \cdot \left[A e^{i\sigma} + B e^{-i\sigma} \right].$

$$= e^{1} \, \cancel{A} \, A \left[e^{i\sigma} - e^{-i\sigma} \right]$$

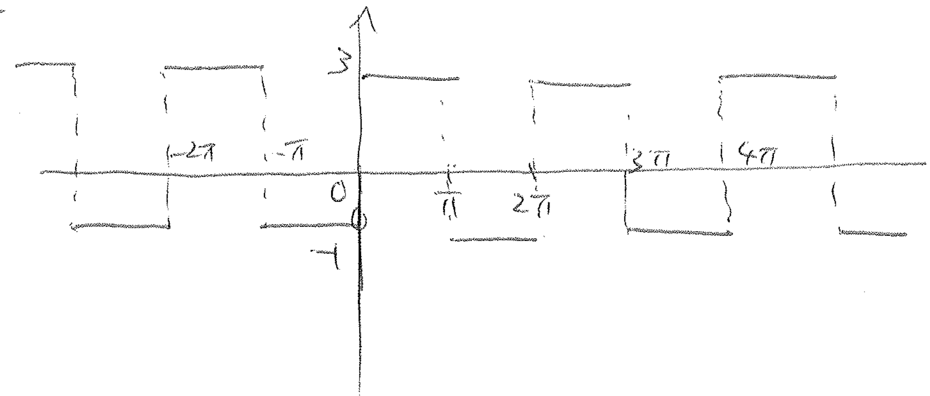
$$= 2ie A \sin\sigma$$

For nontrivial sol., $A \neq 0$.

$\therefore \sin\sigma = 0 \quad \therefore \boxed{\sigma = k\pi \quad k = 1, 2, 3, \cdots \quad (\text{since } \sigma > 0)}$

Prob 6 (4)

A:



B:

period $= 2\pi = 2L \qquad L = \pi$.

$$\hat{f}[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ik\pi t/\pi}\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} -1 \cdot e^{-ikt}\, dt + \frac{1}{2\pi}\int_{0}^{\pi} 3\, e^{-ikt}\, dt$$

$k=0$: $\hat{f}[0] = \frac{1}{2\pi}\cdot(-\pi) + \frac{1}{2\pi}\times 3\pi = 1$

$k\neq 0$: $\hat{f}[k] = \frac{1}{2\pi}\left[\frac{e^{-ikt}}{ik}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[3\cdot\frac{e^{-ikt}}{-ik}\right]_{0}^{\pi}$

$$= \frac{1}{2\pi}\left\{\frac{1}{ik}\left[1 - e^{ik\pi}\right] + \frac{3}{ik}\left(1 - e^{-ik\pi}\right)\right\}$$

$$= \frac{1}{2\pi(ik)}\cdot 4\left(1 - e^{ik\pi}\right)$$

$\longleftarrow$ note $e^{ik\pi} = e^{-ik\pi} = (-1)^k$.

$$= \frac{2}{\pi\cdot ik}\cdot\left[1 - (-1)^k\right]$$

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C:

$$J = \sum_{k=-\infty}^{\infty} \hat{f}[k] \, (-1)^k$$

$$= \sum_{k=-\infty}^{\infty} \hat{f}[k] \, e^{ik\pi}$$

$$= \sum_{k=-\infty}^{\infty} \hat{f}[k] \, e^{ik \cdot \frac{\pi}{\pi} \times \pi}$$

$t = \pi$

& $e^{ik\frac{\pi t}{L}}$

(with $L = \pi$)

$$= \frac{1}{2}\left(f(\pi-) + f(\pi+) \right)$$

$$= \frac{1}{2}\left(3 + (-1)\right) = 1$$

Prob 2

A: From Formula sheet: $\frac{1}{2\pi} \operatorname{sinc}\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} \operatorname{rect}(\omega)$

$\therefore \operatorname{sinc}\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \operatorname{rect}(\omega)$

Scaling $\operatorname{sinc}(t) \xrightarrow{\mathcal{F}} 2\pi \cdot \frac{1}{2} \operatorname{rect}\left(\frac{\omega}{2}\right)$

time-shift $\operatorname{sinc}(t-2) \xrightarrow{\mathcal{F}} e^{-i\omega 2} \cdot 2\pi \frac{1}{2} \operatorname{rect}\left(\frac{\omega}{2}\right)$

$$\therefore \pi e^{-i\omega 2} \operatorname{rect}\left(\frac{\omega}{2}\right)$$

B: Parseval:

$$\int_{-\infty}^{\infty} |\mathrm{sinc}(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt$$

for $\mathcal{F}[x(t)](\omega) = \mathrm{sinc}(\omega)$

Note $\mathrm{sinc}\left(\frac{\omega}{2}\right) \xrightarrow{\mathcal{F}^{-1}} \mathrm{rect}(t)$

scaling: $\cancel{\frac{1}{2}\mathrm{sinc}}$

$$2\,\mathrm{sinc}\left(\frac{\omega}{2} \times 2\right) \xrightarrow{\mathcal{F}^{-1}} \mathrm{rect}\left(\frac{t}{2}\right)$$

$$\therefore \mathrm{sinc}(\omega) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right)$$

$$\therefore x(t) = \frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right)$$

$$\therefore \int_{-\infty}^{\infty} |\mathrm{sinc}(\omega)|^2 d\omega$$

$$= 2\pi \int_{-\infty}^{\infty} \left|\frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right)\right|^2 dt$$

$$= 2\pi \times \frac{1}{4} \int_{-\infty}^{\infty} \left|\mathrm{rect}\left(\frac{t}{2}\right)\right|^2 dt = \frac{\pi}{2} \cdot \int_{-1}^{1} dt$$

$$= \pi$$

$\square$

Prob 8

First consider

$$\frac{i\omega+2}{(i\omega+1)(i\omega+3)}$$

$$= \frac{A}{i\omega+1} + \frac{B}{i\omega+3}$$

$$\left(\begin{array}{c} A+B=1, \quad 3A+B=2 \\ A=\frac{1}{2}, \quad B=\frac{1}{2} \end{array} \right)$$

$$= \frac{1}{2}\cdot\frac{1}{i\omega+1} + \frac{1}{2}\frac{1}{i\omega+3}$$

$$\xrightarrow{\mathcal{F}^{-1}} \frac{1}{2}\cdot e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

$$= \frac{1}{2}\left(e^{-t}+e^{-3t}\right)u(t)$$

Now, frequency shift

$$\frac{i(\omega-1)+2}{(i(\omega-1)+1)(i(\omega-1)+3)} \xrightarrow{\mathcal{F}^{-1}} e^{it}\left[\frac{1}{2}\left(e^{-t}+e^{-3t}\right)u(t)\right]$$

time-shift

$$e^{-i3\omega}\left(\frac{i(\omega-1)+2}{(i(\omega-1)+2)(i(\omega-1)+3)}\right) \xrightarrow{\mathcal{F}^{-1}} e^{i(t-3)}\left[\frac{1}{2}\left(e^{-(t-3)}+e^{-3(t-3)}\right)\right] \times u(t-3).$$

$\therefore$ Answer : $\boxed{\frac{1}{2}e^{i(t-3)}\times\left(e^{-(t-3)}+e^{-3(t-3)}\right)u(t-3)}$

Prob 10

period $N = 4$

$$\hat{x}[k] = \frac{1}{4} \sum_{n=0}^{3} x[n]\, e^{-2\pi i \frac{kn}{4}}$$

$$= \frac{1}{4} \sum_{n=0}^{2} e^{-2\pi i \frac{kn}{4}}$$

$$= \frac{1}{4} \left[\left(\sum_{n=0}^{3} e^{-2\pi i \frac{kn}{4}} \right) - e^{-2\pi i\, k \cdot \cancel{\frac{4}{4}}} \right]$$

$$= \frac{1}{4} \sum_{n=0}^{3} e^{-2\pi i \frac{kn}{4}} - \frac{1}{4}$$

(using orthogonality)

$$= \begin{cases} 1 - \frac{1}{4} & k = 0 \\ 0 - \frac{1}{4} & k = 1, 2, 3 \end{cases}$$

$$= \begin{cases} \frac{3}{4} & k = 0 \\ 0 & k = 1, 2, 3 \end{cases}$$

$\therefore \quad \hat{x}[k] = [\frac{3}{4}, 0, 0, 0]$

Prob 11

N = 6

$$x[n] = \cos\left(\frac{2\pi}{6}n\right) - \sin\left(\frac{2\pi}{6}n\right)$$

$$= \frac{1}{2}\left(e^{i\frac{2\pi}{6}n} + e^{-i\frac{2\pi}{6}n}\right)$$

$$- \frac{1}{2i}\left(e^{i\frac{2\pi}{6}n} - e^{-i\frac{2\pi}{6}n}\right)$$

$$= \left(\frac{1}{2} - \frac{1}{2i}\right) e^{i\frac{2\pi}{6}n} + \left(\frac{1}{2} + \frac{1}{2i}\right) e^{-i\frac{2\pi}{6}n}$$

$$= \sum_{k=0}^{5} \hat{x}[k] \, e^{2\pi i \frac{kn}{6}}$$

$\therefore$ $\hat{x}[1] = \frac{1}{2} - \frac{1}{2i}$

$\hat{x}[-1] = \hat{x}[5] = \frac{1}{2} + \frac{1}{2i}$

$\hat{x}[k] = 0$ & $k = 0, 2, 3, 4$

$$\boxed{\therefore \ \hat{x}[k] = \left[0,\ \frac{1}{2} - \frac{1}{2i},\ 0,\ 0,\ \frac{1}{2} + \frac{1}{2i}\right]}$$

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Prob 12

$$x[n] = \left(\tfrac{1}{3}\right)^n u[n-1]$$

$$\hat{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-i\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{3}\right)^n u[n-1]\, e^{-i\omega n}$$

$$= \sum_{n=1}^{\infty} \left(\tfrac{1}{3}\right)^n e^{-i\omega n}$$

$$= \sum_{m=0}^{\infty} \left(\tfrac{1}{3}\right)^{m+1} e^{-i\omega(m+1)} \quad \longleftarrow (n = m+1)$$

$$= \tfrac{1}{3} \cdot e^{-i\omega} \sum_{m=0}^{\infty} \left(\tfrac{1}{3}\right)^m e^{-i\omega m}$$

$$= \tfrac{1}{3} e^{-i\omega} \sum_{m=0}^{\infty} \underbrace{\left(\tfrac{1}{3} e^{-i\omega}\right)^m} \longleftarrow \sum_{m=0}^{\infty} r^m$$

$$|r| = \tfrac{1}{3} < 1$$

$$= \boxed{\tfrac{1}{3} e^{-i\omega} \frac{1}{1 - \frac{1}{3} e^{-i\omega}}}$$

(11)

Prob 13

$$2^n u[n] \xrightarrow{\mathcal{Z}} \frac{z}{z-2} \quad ; \quad |z| > 2$$

$$-3^n u[-n-1] \xrightarrow{\mathcal{Z}} \frac{z}{z-3} \quad ; \quad |z| < 3$$

$$\therefore \quad 2^n u[n] - 3^n u[-n-1] \xrightarrow{\mathcal{Z}} \frac{z}{z-2} + \frac{z}{z-3} \; ; \; \underbrace{2 < |z| < 3}_{\text{ROC}}$$

Prob 14

$$H(z) = \frac{1}{z^6 - 4z^4}$$

$$= z^{-4} \frac{1}{z^2 - 4}$$

$$= z^{-4} \frac{1}{(z+2)(z-2)}$$

$$= \frac{1}{4} z^{-4} \left(\frac{1}{z-2} - \frac{1}{z+2} \right)$$

$$= \frac{1}{4} z^{-5} \left(\frac{z}{z-2} - \frac{z}{z-(-2)} \right) \qquad \text{ROC: } |z| > 2$$

$$\frac{z}{z-2} \; ; \; \text{ROC } |z| > 2 \xrightarrow{\mathcal{Z}^{-1}} 2^n u[n]$$

$$\frac{z}{z-(-2)} \; ; \; \text{ROC } |z| > 2 \xrightarrow{\mathcal{Z}^{-1}} (-2)^n u[n]$$

$\therefore$

$$\frac{z}{z-2} + \frac{z}{z-(-2)} \quad ; \text{ ROC } |z| > 2$$

$$\xrightarrow{\mathcal{Z}^{-1}} \quad 2^n u[n] + (-2)^n u[n]$$

$$= \left(2^n + (-2)^n\right) u[n]$$

$$z^{-5}\left(\frac{z}{z-2} + \frac{z}{z-(-2)}\right) \quad ; \text{ ROC } |z| > 2$$

$$\underset{\text{(by time-shift)}}{\xrightarrow{\mathcal{Z}^{-1}}} \quad \left(2^{n-5} + (-2)^{n-5}\right) u[n-5]$$

$\therefore$ Final Answer $\boxed{\frac{1}{4}\left[2^{n-5} + (-2)^{n-5}\right] u[n-5]}$

Prob 15 For $x[n] = \delta[n]$.

A: we have

$$h[n] = h[n+1] - 2h[n-1] + \delta[n]$$

$$\xrightarrow{\mathcal{Z}} \quad H(z) = zH(z) - 2z^{-1}H(z) + 1.$$

$$(1 - z + 2z^{-1})\,H(z) = 1$$

$$\therefore \quad H(z) = \frac{1}{1 - z + 2z^{-1}}$$

$$= \frac{z}{z - z^2 + 2} = -\frac{z}{z^2 - z - 2}$$

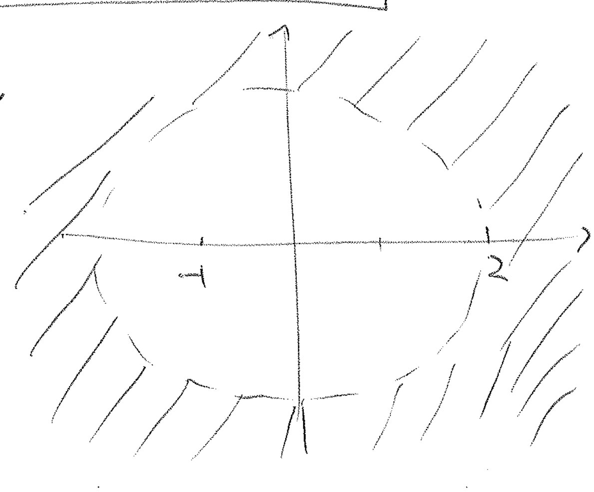
$$= -z \cdot \frac{1}{(z-2)(z+1)}$$

$$= \frac{-z}{3}\left(\frac{1}{z-2} - \frac{1}{z+1}\right)$$

$$\therefore \quad \boxed{H(z) = -\frac{1}{3}\left(\frac{z}{z-2} - \frac{z}{z+1}\right).}$$

Since this LTI is causal,

ROC : $|z| > 2$



By causality, we have to choose this ROC

B:

$$\frac{z}{z+1} \; ; \; \text{ROC: } |z|>1 \quad \xrightarrow{\mathcal{Z}^{-1}} \quad (-1)^n u[n]$$

$$\frac{z}{z-2} \; ; \; \text{ROC: } |z|>2 \quad \xrightarrow{\mathcal{Z}^{-1}} \quad 2^n u[n]$$

$$\therefore \; -\frac{1}{3}\left(\frac{z}{z-2} - \frac{z}{z+1}\right) \xrightarrow{\mathcal{Z}^{-1}} h[n] = -\frac{1}{3}\left(2^n u[n] - (-1)^n u[n]\right)$$

$$= -\frac{1}{3}\left(2^n - (-1)^n\right) u[n]$$