Be sure this exam has 11 pages including the cover

The University of British Columbia

Final Exam – December 2009
Mathematics 267, Mathematical Methods for EE and CS Students

Name ______________________ **Signature** ______________________

Student Number ______________________

This exam consists of **5** questions worth **100** marks in total. No notes, calculators aids are permitted.

Problem	max score	score
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
total	100	

(20 points) 1. Find the solution $u(x,t)$ of the wave equation:

$$\begin{cases} u_{tt}(x,t) = \frac{1}{\pi^2}u_{xx}(x,t), & 0 \le x \le 1, \quad t > 0, \\ u_t(x,0) = 0, & 0 \le x \le 1, \\ u(x,0) = f(x), & 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, & t \ge 0. \end{cases}$$

Here

$$f(x) = \begin{cases} \frac{3}{10}x, & \text{if } 0 \le x \le \frac{1}{3}, \\ \frac{3(1-x)}{20}, & \text{if } \frac{1}{3} \le x \le 1. \end{cases}$$

From formula sheet with $L=1$, $c = \frac{1}{\pi^2}$ and $g(x)=0$

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{nt}{\pi}\right) + B_n \sin\left(\frac{nt}{\pi}\right)\right] \sin(n\pi x)$$

with $B_n = 0$ and

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2}\right)dx = 2\int_0^1 f(x)\sin(n\pi x)dx$$

$$= 2\int_0^{1/3} \frac{3}{10}x\sin(n\pi x)dx + 2\int_{1/3}^1 \frac{3}{20}(1-x)\sin(n\pi x)dx$$

$$= I_1 + I_2$$

where,

$$I_1 = \frac{3}{5}\int_0^{1/3} x\sin(n\pi x)\,dx = \ldots$$ integration by parts

$$= -\frac{1}{5}\frac{1}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{3}{5}\frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)$$

and,

$$I_2 = \frac{3}{10}\int_{1/3}^{1}(1-x)\sin(n\pi x)\,dx = \ldots$$ integration by parts

$$= \frac{1}{5}\,\frac{1}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{3}{10}\,\frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)$$

$$\Rightarrow A_n = I_1 + I_2 = \frac{9}{10}\cdot\frac{1}{n^2\pi^2}\cdot\sin\left(\frac{n\pi}{3}\right)$$

$$\Rightarrow u(x,t) = \frac{9}{10}\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{nt}{\pi}\right)\sin(n\pi x)$$

(20 points) 2. Let $g(x)$ be the 2π-periodic triangle wave and

$$g(x) = \begin{cases} \pi + x, & \text{if } -\pi \le x \le 0, \\ \pi - x, & \text{if } 0 \le x \le \pi. \end{cases}$$

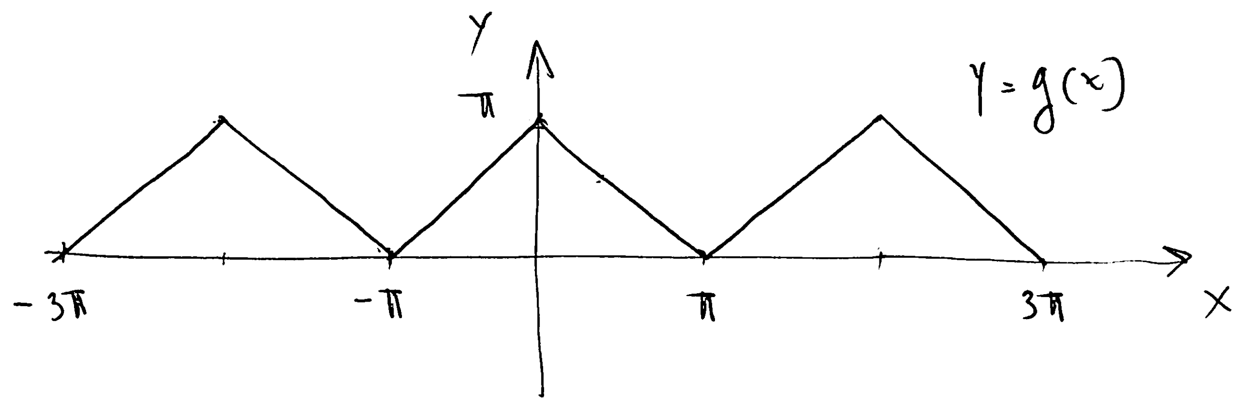
(a) Plot the graph of $g(x)$ in at least three period.

(b) Find the real Fourier series of $g(x)$.

(c) Find

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

a)



b) Notice: $g(x)$ is an even function,
thus we can simply compute the cosine fourier series.

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

with $L = \pi$ and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,dx$$

$$= ,,, = -\frac{2}{\pi}\frac{1}{n^2}\left[(-1)^n - 1\right]$$

integration by parts

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{2}{\pi}\left(\pi x - \frac{x^2}{2}\right)\Big|_0^{\pi} = \pi$$

$$\Rightarrow \quad a_n = \begin{cases} \pi, & n = 0 \\ \frac{4}{\pi n^2}, & n \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad g(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\cos(nx)}{n^2}$$

$$\text{or} \quad g(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2},$$

(20 points) 3. For the following questions, fill in the answers in the boxes. No work need to be shown and no partial credit will be given. **Anything written outside of the boxes is ignored.**

(a) Find the fundamental period of the signal $x[n] = 3 + \cos(\frac{5\pi n}{7} + \frac{\pi}{3})$.

Answer = $\boxed{}$

time (DTFT)

(b) Find the discrete Fourier transform of $x[n] = (\frac{1}{4})^n u(n-3)$.

Answer = $\boxed{\hat{x}(w)}$

$$\hat{x}(w) = \frac{1}{64} \cdot \frac{e^{-i3w}}{1 - \frac{1}{4}e^{-iw}}$$

see back

(c) Find the Fourier transform of $f(t) = te^{-3t}u(t-3)$.

Answer = $\boxed{\hat{f}(w)}$

$$\hat{f}(w) = e^{-9} e^{-i3w} \frac{10 + i3w}{(3+iw)^2}$$

see back

b) $x[n] = \left(\frac{1}{4}\right)^n u[n-3] = \left(\frac{1}{4}\right)^3 \left(\frac{1}{4}\right)^{n-3} u[n-3]$

let $y[n] = \left(\frac{1}{4}\right)^n u[n]$

$\Rightarrow x[n] = \frac{1}{64} y[n-3]$

$\Rightarrow \hat{x}(w) = \frac{1}{64} e^{-i3w} \hat{y}(w) = \frac{1}{64} e^{-i3w} \cdot \frac{1}{1-\frac{1}{4}e^{-iw}}$

Notice: $\hat{y}(w) = Y(z)\Big|_{z=e^{iw}} = \frac{1}{1-\frac{1}{4}e^{-iw}}$ from table for z-transform

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c) $f(t) = te^{-3t} u(t-3) = e^{-9}\, te^{-3(t-3)} u(t-3)$

let $g(t) = e^{-3t} u(t)$ and $x(t) = g(t-3)$

$\Rightarrow f(t) = e^{-9} t\, g(t-3) = e^{-9} t\, x(t)$

Recall: $t x(t) \longleftrightarrow i \frac{d}{dw} \hat{x}(w)$

$\Rightarrow \hat{f}(w) = e^{-9}\, i \frac{d}{dw} \hat{x}(w)$

where, $\hat{x}(w) = e^{-i3w} \hat{g}(w) = e^{-i3w} \cdot \frac{1}{3+iw}$

$\Rightarrow \hat{f}(w) = e^{-9}\, i \frac{d}{dw}\left[\frac{e^{-i3w}}{3+iw}\right] = e^{-9} e^{-i3w} \frac{10+i3w}{(3+iw)^2}$.

(d) Find the z-transform of $x[n] = \delta[n] + 2^n u[-n]$.

Answer = $\boxed{1 + \frac{2}{2-z},}$

ROC : $\frac{1}{2} < |z| < 2$

see back

(e) Find the inverse Fourier transform of $\hat{f}(\omega) = \frac{1}{1+\omega^2}$.

Answer = $\boxed{\frac{1}{2} e^{-|t|}}$

From the table

$$\underbrace{\frac{1}{2} e^{-|t|}}_{f(t)} \longleftrightarrow \frac{1}{2} \cdot \frac{2}{1+w^2} = \underbrace{\frac{1}{1+w^2}}_{\hat{f}(w)}$$

d) $x[n] = \delta[n] + 2^n u[-n]$

$= \delta[n] + \left(\frac{1}{2}\right)^{-n} u[-n]$

let $y[n] = \left(\frac{1}{2}\right)^n u[n]$

$\Rightarrow x[n] = \delta[n] + y[-n]$

from table:

$$\delta[n] \longleftrightarrow 1, \qquad |z| > 0$$

and $$y[-n] \longleftrightarrow Y\left(\frac{1}{z}\right), \quad \left\{\frac{1}{z} : z \in R_y\right\}$$

where, $$Y(z) = \frac{z}{z - \frac{1}{2}}, \qquad |z| > \frac{1}{2}$$

$$\Rightarrow X(z) = 1 + \frac{1/z}{1/z - 1/2} = 1 + \frac{2}{2-z}, \quad \frac{1}{2} < |z| < 2$$

(20 points) 4. Consider the system with input $x(t)$ and output $y(t)$ which is characterized by the ODE

$$y''(t) + 5y'(t) + 4y(t) = 3x(t).$$

(a) Find the transfer function $\hat{H}(\omega)$ and the impulse response $H(t)$.

(b) Find $y(t)$ if $x(t) = \delta(t-4)$.

(c) Find $y(t)$ if $x(t) = e^{-4t}u(t-2)$.

a) $$[(iw)^2 + 5(iw) + 4]\,\hat{Y}(w) = 3\,\hat{X}(w)$$

$$\Rightarrow \hat{Y}(w) = \frac{3}{(iw)^2 + 5(iw) + 4}\,\hat{X}(w)$$

$$\Rightarrow \hat{H}(w) = \frac{3}{(iw)^2 + 5(iw) + 4} = \frac{3}{(1+iw)(4+iw)}$$

$$= \frac{A}{1+iw} + \frac{B}{4+iw}$$

$$\Rightarrow A = \left.\frac{3}{4+iw}\right|_{iw=-1} = \frac{3}{4-1} = 1$$

$$B = \left.\frac{3}{1+iw}\right|_{iw=-4} = \frac{3}{1-4} = -1$$

$$\Rightarrow \hat{H}(w) = \frac{1}{1+iw} - \frac{1}{4+iw}$$

$$\Rightarrow H(t) = e^{-t}u(t) - e^{-4t}u(t)$$
$$= (e^{-t} - e^{-4t})\,u(t)$$

b) $Y(t) = (H * \delta_4)(t) = H(t-4)$
$$= \left(e^{-(t-4)} - e^{-4(t-4)}\right)\cdot u(t-4)$$

c) $x(t) = e^{-4t}u(t-2) = e^{-8}\,e^{-4(t-2)}\,u(t-2)$

let $z(t) = e^{-4t}\,u(t)$

$$\Rightarrow x(t) = e^{-8}\,z(t-2)$$
$$\Rightarrow \hat{x}(w) = e^{-8}\,e^{-i2w}\,\hat{z}(w) = e^{-8}\,e^{-i2w}\cdot\frac{1}{4+iw}$$

recall: $\hat{y}(w) = \hat{H}(w)\,\hat{x}(w)$

$$\Rightarrow \hat{Y}(w) = e^{-8}\,e^{-i2w}\cdot\frac{3}{(1+iw)(4+iw)^2}$$

partial fractions

$$= e^{-8}\,e^{-i2w}\,\frac{1}{3}\left[\frac{1}{1+iw} - \frac{7+iw}{(4+iw)^2}\right]$$

$$\Rightarrow \hat{y}(w) = \frac{1}{3}e^{-8}e^{-i2w}\left[\frac{1}{1+iw} - \frac{1}{4+iw} - \frac{3}{(4+iw)^2}\right]$$

Notice:
$$\frac{3}{(4+iw)^2} = i\frac{d}{dw}\left[\frac{3}{4+iw}\right]$$

$$\Rightarrow \hat{y}(w) = \frac{1}{3}e^{-8}e^{-i2w}\underbrace{\left[\frac{1}{1+iw} - \frac{1}{4+iw} - i\frac{d}{dw}\left(\frac{3}{4+iw}\right)\right]}_{\hat{z}(w)}$$

$$\Rightarrow \hat{y}(w) = \frac{1}{3}e^{-8}e^{-i2w}\hat{z}(w)$$

$$\Rightarrow y(t) = \frac{1}{3}e^{-8}\, z(t-2),$$

with $z(t) = e^{-t}u(t) - e^{-4t}u(t) - 3te^{-4t}u(t)$

$$\Rightarrow y(t) = \frac{1}{3}e^{-8}\left[e^{-(t-2)} - e^{-4(t-2)} - 3(t-2)e^{-4(t-2)}\right]u(t-2)$$

or,

$$y(t) = e^{-8}\left[\frac{1}{3}e^{-(t-2)} - \left(t-\frac{5}{3}\right)e^{-4(t-2)}\right]u(t-2),$$