

Be sure this exam has 11 pages including the cover

The University of British Columbia

Final Exam – December 2009
Mathematics 267, Mathematical Methods for EE and CS Students

Name _____ Signature _____

Student Number _____

This exam consists of 5 questions worth 100 marks in total. No notes, calculators aids are permitted.

Problem	max score	score
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
total	100	

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

(20 points) 1. Find the solution $u(x, t)$ of the wave equation:

$$\begin{cases} u_{tt}(x, t) = \frac{1}{\pi^2} u_{xx}(x, t), & 0 \leq x \leq 1, \quad t > 0, \\ u_t(x, 0) = 0, & 0 \leq x \leq 1, \\ u(x, 0) = f(x), & 0 \leq x \leq 1, \\ u(0, t) = u(1, t) = 0, & t \geq 0. \end{cases}$$

Here

$$f(x) = \begin{cases} \frac{3}{10}x, & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{3(1-x)}{20}, & \text{if } \frac{1}{3} \leq x \leq 1. \end{cases}$$

From formula sheet with $L=1$, $c = \frac{1}{\pi^2}$ and $g(x) = 0$

$$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n t}{\pi}\right) + B_n \sin\left(\frac{n t}{\pi}\right) \right] \sin(n \pi x)$$

with $B_n = 0$ and

$$\begin{aligned} A_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n \pi x}{L}\right) dx = 2 \int_0^1 f(x) \sin(n \pi x) dx \\ &= 2 \int_0^{1/3} \frac{3}{10} x \sin(n \pi x) dx + 2 \int_{1/3}^1 \frac{3}{20} (1-x) \sin(n \pi x) dx \\ &= I_1 + I_2 \end{aligned}$$

where,

$$\begin{aligned} I_1 &= \frac{3}{5} \int_0^{1/3} x \sin(n \pi x) dx = \dots \quad \text{integration by parts} \\ &= -\frac{1}{5} \frac{1}{n \pi} \cos\left(\frac{n \pi}{3}\right) + \frac{3}{5} \frac{1}{n^2 \pi^2} \sin\left(\frac{n \pi}{3}\right) \end{aligned}$$

and,

$$I_2 = \frac{3}{10} \int_{1/3}^1 (1-x) \sin(n\pi x) dx = \dots \quad \text{integration by parts}$$

$$= \frac{1}{5} \frac{1}{n\pi} \cos\left(\frac{n\pi}{3}\right) + \frac{3}{10} \frac{1}{n^2\pi^2} \sin\left(\frac{n\pi}{3}\right)$$

$$\Rightarrow A_n = I_1 + I_2 = \frac{9}{10} \cdot \frac{1}{n^2\pi^2} \cdot \sin\left(\frac{n\pi}{3}\right)$$

$$\Rightarrow u(x,t) = \frac{9}{10} \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \sin\left(\frac{n\pi}{3}\right) \cos\left(\frac{nt}{\pi}\right) \sin(n\pi x)$$

(20 points) 2. Let $g(x)$ be the 2π -periodic triangle wave and

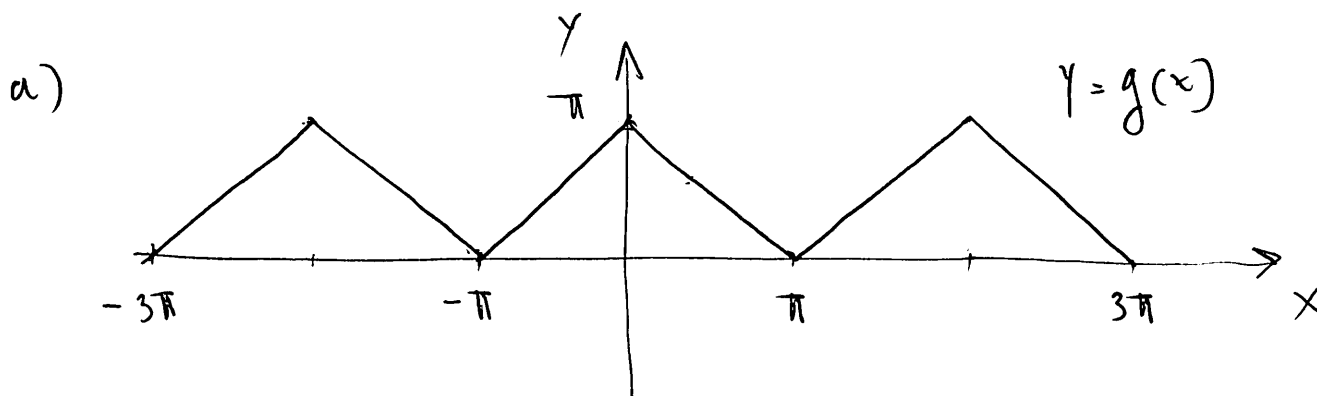
$$g(x) = \begin{cases} \pi + x, & \text{if } -\pi \leq x \leq 0, \\ \pi - x, & \text{if } 0 \leq x \leq \pi. \end{cases}$$

(a) Plot the graph of $g(x)$ in at least three period.

(b) Find the real Fourier series of $g(x)$.

(c) Find

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$



b) Notice: $g(x)$ is an even function,
thus we can simply compute the cosine Fourier series.

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

with $L = \pi$ and

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos(nx) dx \\ &= \dots = -\frac{2}{\pi} \frac{1}{n^2} [(-1)^n - 1] \end{aligned}$$

integration
by parts

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{2}{\pi} \left(\pi x - \frac{x^2}{2} \right) \Big|_0^{\pi} = \pi$$

$$\Rightarrow a_n = \begin{cases} \pi, & n=0 \\ \frac{4}{\pi n^2}, & n \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\cos(nx)}{n^2}$$

$$\text{or } f(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}$$

(20 points) 3. For the following questions, fill in the answers in the boxes. No work need to be shown and no partial credit will be given. Anything written outside of the boxes is ignored.

(a) Find the fundamental period of the signal $x[n] = 3 + \cos(\frac{5\pi n}{7} + \frac{\pi}{3})$.

Answer =

time
↓
(DTFT)

(b) Find the discrete Fourier transform of $x[n] = (\frac{1}{4})^n u(n-3)$.

Answer =

$$\hat{X}(w) = \frac{1}{64} \cdot \frac{e^{-i3w}}{1 - \frac{1}{4}e^{-iw}}$$

see back

(c) Find the Fourier transform of $f(t) = te^{-3t}u(t-3)$.

Answer =

$$\hat{f}(w) = e^{-9} e^{-i3w} \frac{10 + i3w}{(3 + iw)^2}$$

see back

$$b) x[n] = \left(\frac{1}{4}\right)^n u[n-3] = \left(\frac{1}{4}\right)^3 \left(\frac{1}{4}\right)^{n-3} u[n-3]$$

$$\text{let } y[n] = \left(\frac{1}{4}\right)^n u[n]$$

$$\Rightarrow x[n] = \frac{1}{64} y[n-3]$$

$$\Rightarrow \hat{x}(\omega) = \frac{1}{64} e^{-i3\omega} \hat{y}(\omega) = \frac{1}{64} e^{-i3\omega} \cdot \frac{1}{1 - \frac{1}{4} e^{-i\omega}}$$

Notice: $\hat{y}(\omega) = Y(z) \Big|_{z=e^{i\omega}} = \frac{1}{1 - \frac{1}{4} e^{-i\omega}}$

from table
for z-transform

$$c) f(t) = t e^{-3t} u(t-3) = e^{-9} t e^{-3(t-3)} u(t-3)$$

$$\text{let } g(t) = e^{-3t} u(t) \text{ and } x(t) = g(t-3)$$

$$\Rightarrow f(t) = e^{-9} t g(t-3) = e^{-9} t x(t)$$

Recall: $t x(t) \longleftrightarrow i \frac{d}{d\omega} \hat{x}(\omega)$

$$\Rightarrow \hat{f}(\omega) = e^{-9} i \frac{d}{d\omega} \hat{x}(\omega)$$

$$\text{where, } \hat{x}(\omega) = e^{-i3\omega} \hat{g}(\omega) = e^{-i3\omega} \cdot \frac{1}{3 + i\omega}$$

$$\Rightarrow \hat{f}(\omega) = e^{-9} i \frac{d}{d\omega} \left[\frac{e^{-i3\omega}}{3 + i\omega} \right] = e^{-9} e^{-i3\omega} \frac{10 + i3\omega}{(3 + i\omega)^2}$$

(d) Find the z-transform of $x[n] = \delta[n] + 2^n u[-n]$.

Answer = $\boxed{1 + \frac{2}{2-z}}$

RBC : $\frac{1}{2} < |z| < 2$

see back

(e) Find the inverse Fourier transform of $\hat{f}(\omega) = \frac{1}{1+\omega^2}$.

Answer = $\boxed{\frac{1}{2} e^{-|t|}}$

from the table

$$\underbrace{\frac{1}{2} e^{-|t|}}_{\hat{f}(t)} \longleftrightarrow \frac{1}{2} \cdot \frac{2}{1+\omega^2} = \underbrace{\frac{1}{1+\omega^2}}_{\hat{f}(\omega)}$$

$$d) \quad x[n] = \delta[n] + 2^n u[-n]$$

$$= \delta[n] + \left(\frac{1}{2}\right)^{-n} u[-n]$$

$$\text{let } y[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$\Rightarrow x[n] = \delta[n] + y[-n]$$

from table:

$$\delta[n] \longleftrightarrow 1, \quad |z| > 0$$

$$\text{and } y[n] \longleftrightarrow Y\left(\frac{1}{z}\right), \quad \left\{ \frac{1}{z} : z \in R_y \right\}$$

$$\text{where, } Y(z) = \frac{z}{z - \frac{1}{2}}, \quad |z| > \frac{1}{2}$$

$$\Rightarrow X(z) = 1 + \frac{1/z}{1/z - 1/2} = 1 + \frac{z}{2-z}, \quad \frac{1}{2} < |z| < 2$$

(20 points) 4. Consider the system with input $x(t)$ and output $y(t)$ which is characterized by the ODE

$$y''(t) + 5y'(t) + 4y(t) = 3x(t).$$

(a) Find the transfer function $\hat{H}(\omega)$ and the impulse response $H(t)$.

(b) Find $y(t)$ if $x(t) = \delta(t - 4)$.

(c) Find $y(t)$ if $x(t) = e^{-4t}u(t - 2)$.

$$a) \quad [(\bar{i}\omega)^2 + 5(\bar{i}\omega) + 4] \hat{Y}(\omega) = 3 \hat{X}(\omega)$$

$$\Rightarrow \hat{Y}(\omega) = \frac{3}{(\bar{i}\omega)^2 + 5(\bar{i}\omega) + 4} \hat{X}(\omega)$$

$$\Rightarrow \hat{H}(\omega) = \frac{3}{(\bar{i}\omega)^2 + 5(\bar{i}\omega) + 4} = \frac{3}{(1 + \bar{i}\omega)(4 + \bar{i}\omega)}$$

$$= \frac{A}{1 + \bar{i}\omega} + \frac{B}{4 + \bar{i}\omega}$$

$$\Rightarrow A = \frac{3}{4 + \bar{i}\omega} \Big|_{\bar{i}\omega = -1} = \frac{3}{4 - 1} = 1$$

$$B = \frac{3}{1 + \bar{i}\omega} \Big|_{\bar{i}\omega = -4} = \frac{3}{1 - 4} = -1$$

$$\Rightarrow \hat{H}(\omega) = \frac{1}{1 + \bar{i}\omega} - \frac{1}{4 + \bar{i}\omega}$$

$$\Rightarrow H(t) = e^{-t} u(t) - e^{-4t} u(t) \\ = (e^{-t} - e^{-4t}) u(t)$$

$$b) Y(t) = (H * \delta_4)(t) = H(t-4) \\ = (e^{-(t-4)} - e^{-4(t-4)}) \cdot u(t-4)$$

$$c) x(t) = e^{-4t} u(t-2) = e^{-8} e^{-4(t-2)} u(t-2)$$

$$\text{let } z(t) = e^{-4t} u(t)$$

$$\Rightarrow x(t) = e^{-8} z(t-2)$$

$$\Rightarrow \hat{x}(w) = e^{-8} e^{-i2w} \hat{z}(w) = e^{-8} e^{-i2w} \cdot \frac{1}{4+iw}$$

$$\text{recall: } \hat{y}(w) = \hat{H}(w) \hat{x}(w)$$

$$\Rightarrow \hat{y}(w) = e^{-8} e^{-i2w} \cdot \frac{3}{(1+iw)(4+iw)^2}$$

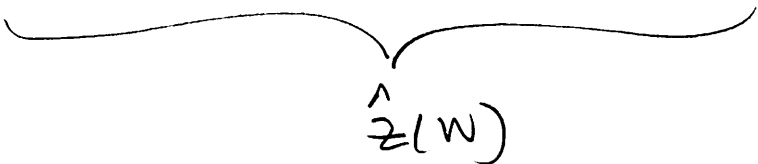
partial fractions

$$= e^{-8} e^{-i2w} \frac{1}{3} \left[\frac{1}{1+iw} - \frac{7+iw}{(4+iw)^2} \right]$$

$$\Rightarrow \hat{y}(w) = \frac{1}{3} e^{-8} e^{-i2w} \left[\frac{1}{1+iw} - \frac{1}{4+iw} - \frac{3}{(4+iw)^2} \right]$$

Notiu : $\frac{3}{(4+i\omega)^2} = i \frac{d}{d\omega} \left[\frac{3}{4+i\omega} \right]$

$$\Rightarrow \hat{Y}(\omega) = \frac{1}{3} e^{-8} e^{-i2\omega} \left[\frac{1}{1+i\omega} - \frac{1}{4+i\omega} - i \frac{d}{d\omega} \left(\frac{3}{4+i\omega} \right) \right]$$



 $\hat{Z}(\omega)$

$$\Rightarrow \hat{Y}(\omega) = \frac{1}{3} e^{-8} e^{-i2\omega} \hat{Z}(\omega)$$

$$\Rightarrow Y(t) = \frac{1}{3} e^{-8} Z(t-2),$$

with $Z(t) = e^{-t} u(t) - e^{-4t} u(t) - 3te^{-4t} u(t)$

$$\Rightarrow Y(t) = \frac{1}{3} e^{-8} \left[e^{-(t-2)} - e^{-4(t-2)} - 3(t-2)e^{-4(t-2)} \right] u(t-2)$$

or,

$$Y(t) = e^{-8} \left[\frac{1}{3} e^{-(t-2)} - \left(t - \frac{5}{3}\right) e^{-4(t-2)} \right] u(t-2),$$