Math 267 Midterm I

Oct 13, 2010

Duration: 50 minutes

Name: ________________ Student Number: ________________

This exam should have 9 pages, including this cover sheet. No textbooks, calculators, or other aids are allowed. Double-click to edit.
There are 4 questions in this exam, each worth 10 points.

**Problem 1 (10 points)**

(a) Calculate $ie^{i\pi/2}+1$.

(b) Suppose $e^{i\theta}$ is a complex number with $\mathrm{Re}\{e^{i\theta}\}>0$ and $\mathrm{Im}\{e^{i\theta}\}=\sqrt{5}/5$.

(i) Find $\mathrm{Re}\{e^{i\theta}\}$.

(ii) Calculate $\cos(2\theta)$.

(c) Calculate $\displaystyle\sum_{n=0}^{100}\left[\cos\left(\frac{2\pi}{100}n\right)+i\sin\left(\frac{2\pi}{100}n\right)\right]$.

SOLUTION

(a) $e^{i\pi/2}=i \Rightarrow i.e^{i\frac{\pi}{2}}+1 = i.i+1 = i^2+1 = \underline{\underline{0}}$

(b)(i) Let $\mathrm{Re}\{e^{i\theta}\}=x$. Then

$$e^{i\theta}=x+i\frac{\sqrt{5}}{5}.$$

We know that $|e^{i\theta}|=\sqrt{x^2+\frac{1}{5}}=1$

$\Rightarrow x^2=\frac{4}{5} \Rightarrow x=\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$ (we used the fact that $x>0$).

(ii) See next page.

(ii) Note that $\left(e^{i\theta}\right)^2 = e^{i2\theta} = \cos 2\theta + i \sin 2\theta$

$$\Rightarrow \quad \cos 2\theta = Re\left\{\left(e^{i\theta}\right)^2\right\} \qquad (*)$$

On the other hand,

$$\left(e^{i\theta}\right)^2 = \left(\frac{2\sqrt{5}}{5} + i\,\frac{\sqrt{5}}{5}\right)^2 = \frac{4\cdot 5}{25} - \frac{5}{25} + i\,\frac{4\sqrt{5}\cdot\sqrt{5}}{5\cdot 5}$$

$$\Rightarrow \quad \left(e^{i\theta}\right)^2 = \frac{3}{5} + i\,\frac{4}{5}$$

So, using $(*)$, we conclude $\cos 2\theta = \frac{3}{5}$.

(c) Observe that $\cos\left(\frac{2\pi}{100}n\right) + i\sin\left(\frac{2\pi}{100}n\right) = e^{i\frac{2\pi}{100}n}$

Then, we need to calculate

$$\sum_{n=0}^{100} e^{i\frac{2\pi}{100}n} = \sum_{n=0}^{100}\left(e^{i\frac{2\pi}{100}}\right)^n \quad \leftarrow \text{geometric sum!}$$

$$= \frac{\left(e^{i\frac{2\pi}{100}}\right)^{101} - 1}{e^{i\frac{2\pi}{100}} - 1} = \frac{e^{i\frac{2\pi}{100}\cdot 101} - 1}{e^{i\frac{2\pi}{100}} - 1}$$

$$= \frac{e^{i\frac{2\pi}{100}} - 1}{e^{i\frac{2\pi}{100}} - 1} = 1 \quad //$$

Here we used the following fact:

$$e^{i\frac{2\pi}{100}\cdot 101} = e^{i\frac{2\pi}{100}(100+1)} = e^{i\left(2\pi + \frac{2\pi}{100}\right)} = \overset{1}{\cancel{e^{i2\pi}}}\cdot e^{i\frac{2\pi}{100}} = e^{i\frac{2\pi}{100}}.$$

**Problem 2 (10 points)** Consider the function

$$f(t) = \sum_{k=-20}^{20} 2^{-|k|} e^{ik\pi t}.$$

Let $\widehat{f}(k)$ denote the complex Fourier coefficients of $f(t)$.

(a) Specify the values of $\widehat{f}(2)$ and $\widehat{f}(-3)$.

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(c) Is $f(t)$ real valued? Even or odd? Justify your answers.

(d) Calculate $\int_{-\pi}^{\pi} |f(t)|^2 dt$.

(a) $\hat{f}(2) = 2^{-2} = \frac{1}{4}$ ; $\hat{f}(-3) = \frac{1}{8}$.

(b) Recall: fund. period $= 2\ell \Rightarrow$ F.S. $\sum c_k e^{ik\frac{\pi}{\ell}t}$

So in our example above, $\ell = 1 \Rightarrow$ fund. period $= 2\ell = 2$

(c) Since $\hat{f}(k) = \begin{cases} 2^{-|k|} ; & -20 \le k \le 20 \\ 0 & \text{otherwise} \end{cases}$

is both even and real, $f(t)$ is also even & real-valued:

• $f(t)$ is real valued since $\hat{f}(-k) = \overline{\hat{f}(k)} = \hat{f}(k)\ \forall k.$

↑ since real-val.

• $f(t)$ is even since $\hat{f}(-k) = \hat{f}(k)$ for all $k$

(d) ~~We use Parseval's identity~~ $\cancel{\frac{1}{\ell}\int_{-\ell}^{\ell}}$

<u>NOTE</u>: THERE IS AN IMPORTANT TYPO WITH THIS PROBLEM - I INTENDED TO ASK

$$\int_{-1}^{1} |f(t)|^2\,dt, \quad \text{not} \quad \int_{-\pi}^{\pi} |f(t)|^2\,dt.$$

~~So, if you did some~~

Here's the solution of the intended problem: By Parseval's identity, we know that (since $\ell = 1$)

$$I = \int_{-1}^{1} |f(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 = \sum_{k=-20}^{20} \left(2^{-|k|}\right)^2.$$

Note that: $\left(2^{-|k|}\right)^2 = 2^{-2|k|} = \left(\frac{1}{4}\right)^{|k|}$. Then

$$I = \sum_{k=-20}^{20} \left(\frac{1}{4}\right)^{|k|} = \underbrace{\sum_{k=-20}^{-1} \left(\frac{1}{4}\right)^{-k}}_{I_1} + 1 + \underbrace{\sum_{k=1}^{20} \left(\frac{1}{4}\right)^{k}}_{I_2}$$

Next, note that $I_1 = I_2$. Let's calculate this quantity:

$$I_2 = \sum_{k=1}^{20} \left(\frac{1}{4}\right)^k = \sum_{n=0}^{19} \left(\frac{1}{4}\right)^{n+1} = \sum_{n=0}^{19} \left(\frac{1}{4}\right)^n \cdot \frac{1}{4} = \frac{1}{4} \cdot \sum_{n=0}^{19} \left(\frac{1}{4}\right)^n$$

$$= \frac{1}{4} \cdot \frac{1 - \left(\frac{1}{4}\right)^{20}}{1 - 1/4} = \cancel{\frac{1}{4}} \cdot \frac{\cancel{4}}{3} \cdot \left(1 - \left(\frac{1}{4}\right)^{20}\right) = \frac{1}{3}\left(1 - \left(\frac{1}{4}\right)^{20}\right)$$

Then $I = 1 + 2 I_2 = 1 + \frac{2}{3}\left(1 - \left(\frac{1}{4}\right)^{20}\right) = \frac{5}{3} - \frac{2}{3} \cdot \left(\frac{1}{4}\right)^{20}.$

<u>REMARK</u>:

**Problem 3 (10 points)**

Let $u(x,t)$ denote the temperature of a straight bar of length 2 at location $x$ and time $t$. Suppose that the ends of the bar are kept at constant temprature 0. The equation describing the variation of temparature in the bar is

$$\text{(PDE)} \qquad u_{xx} = u_t, \qquad 0 < x < 2, \quad t > 0$$

and the boundary conditions are

$$\text{(BC)} \qquad u(0,t) = u(2,t) = 0 \qquad t \geq 0.$$

(a) If $u(x,t) = X(x)T(t)$ satisfies (PDE), find the equations that must be satisfied by $X(x)$ and $T(t)$.

(b) Find all eigenvalues $\sigma_k$ and eigenfunctions $X_k(x)$.

(c) Assuming we can find a solution using this method, what is the limit of $u(x,t)$ as $t$ goes to infinity?

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**Problem 4 (10 points)**

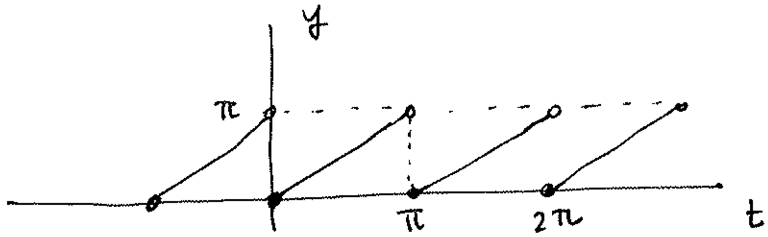
Let $f(t)$ be a $\pi$ periodic function with $f(t) = t$ for $0 \le t < \pi$.

(a) Sketch the graph of $f(t)$ (show at least three periods). Is $f(t)$ odd or even?

(b) Find the complex Fourier series of $f(t)$.

(c) Find the real Fourier series of $f(t)$. i.e. the trigonometric Fourier series.

(a)




$f(t)$ is not odd, and not even.

(b) $\hat{f}(k) = \dfrac{1}{\pi} \displaystyle\int_0^{\pi} \underbrace{t}_{u} \cdot \underbrace{e^{-ik2t}}_{v'}\, dt$ (note: $2\ell = \pi \Rightarrow \ell = \frac{\pi}{2}$)

$\boxed{k \neq 0}$

$$= \frac{1}{\pi}\left[ t \cdot \frac{e^{-ik\cdot 2t}}{-ik2} \Big|_0^{\pi} - \int_0^{\pi} \frac{e^{-ik2t}}{-ik2}\, dt \right]$$

$$= \frac{1}{\pi}\left[ \pi \cdot \frac{e^{-ik2\pi}}{-ik2} - \int_0^{\pi} \frac{e^{-ik2t}}{-ik2}\, dt \right]$$

$$= \frac{1}{\pi}\left[ \frac{\pi}{-ik\cdot 2} - \frac{e^{-ik2t}}{(-ik\cdot 2)^2} \Big|_0^{\pi} \right]$$

$$= \frac{1}{\pi}\left[ \frac{\pi}{-ik\cdot 2} - \left( \frac{e^{-ik2\pi}}{-4k^2} - \frac{1}{-4k^2} \right) \right] = \frac{1}{-ik2} = \frac{i}{2k}.$$

$\boxed{k=0}$ $\qquad \hat{f}(0) = \frac{1}{\pi}\int_0^{\pi} t \, dt = \frac{1}{\pi} \left. \frac{t^2}{2} \right|_0^{\pi} = \pi .$

So: $\hat{f}(k) = \begin{cases} i/2k & k \neq 0 \\ \pi/2 & k = 0 . \end{cases}$ , and

the complex F.S. of $f(t)$ is given by:

$$f(t) = \sum \hat{f}(k) e^{ik\cdot 2 \cdot t}$$ where $\hat{f}(k)$ is as above.

(c) We know that:

$$a_k = \hat{f}(k) + \hat{f}(-k) = \begin{cases} 0 & \text{if } k \neq 0 . \\ \pi & \text{if } k = 0 . \end{cases}$$

$$b_k = i\left(\hat{f}(k) - \hat{f}(-k)\right) = i \cdot \left(\frac{i}{2k} - \frac{i}{(-k)2}\right) = \frac{-1}{k} .$$

So:

$$f(t) = \frac{a_0}{2} + \underbrace{\cancel{\sum_{k=1}^{\infty} a_k \cos(2kt)}}_{\to 0} + \sum_{k=1}^{\infty} b_k \sin(2kt)$$

$$\boxed{f(t) = \pi + \sum -\frac{1}{k} \sin(2kt)}$$