Math 267 Midterm II

Mar 26, 2010

Duration: 50 minutes

Name: ______________________ Student Number: ______________________

This exam should have 8 pages. No textbooks, calculators, or other aids are allowed. One page of notes (two-sided) is allowed. There are 4 questions in this exam, each worth 10 points.

Problem 1 (10 points)

Consider the system with input $x(t)$ and output $y(t)$ characterized by the equation

$$y''(t) + 5y'(t) + 6y(t) = x(t).$$

(a) Find the transfer function $\hat{H}(\omega)$ and the impulse response $H(t)$.

(b) Find $y(t)$ when $x(t) = \delta(t-3)$.

(c) Find $y(t)$ when $x(t) = e^{-t}u(t+4)$.

Solution

(a) Take the Fourier transform of both sides:

$$\left[(i\omega)^2 + 5i\omega + 6\right]\hat{y}(\omega) = \hat{x}(\omega)$$

$$\Rightarrow \boxed{\hat{H}(\omega) = \frac{\hat{y}(\omega)}{\hat{x}(\omega)} = \frac{1}{(i\omega)^2 + 5i\omega + 6}.}$$

Use partial fractions to calculate $H(t)$, the inverse F.T. of $\hat{H}(\omega)$

$$\hat{H}(\omega) = \frac{1}{(i\omega+2)(i\omega+3)} = \frac{1}{i\omega+2} - \frac{1}{i\omega+3}$$

$$\Rightarrow \boxed{H(t) = e^{-2t}u(t) - e^{-3t}u(t) = \left[e^{-2t} - e^{-3t}\right]u(t)}$$

(b) Calculate $y(t)$ in time-domain:

$$\boxed{y(t) = (x * H)(t) = H(t-3) = \left[e^{-2(t-3)} - e^{-3(t-3)}\right]u(t-3)}$$

(c) This time, we first find $\hat{y}(\omega)$:

$$\hat{y}(\omega) = \hat{x}(\omega)\,\hat{H}(\omega).$$

Note that: $x(t) = e^{-t}u(t+4) = e^{4}\cdot e^{-(t+4)}u(t+4)$

$$\Rightarrow \hat{x}(\omega) = e^{4}\cdot e^{i\omega\cdot 4}\frac{1}{i\omega+1}$$

So: $$\hat{y}(\omega) = e^{4}\cdot e^{i\omega 4}\cdot\frac{1}{(i\omega+1)(i\omega+2)(i\omega+3)}$$

$$= e^{4}\cdot e^{i\omega 4}\left[\frac{1/2}{i\omega+1} - \frac{1}{i\omega+2} + \frac{1/2}{i\omega+3}\right]$$

$$\Rightarrow \boxed{y(t) = e^{4}\cdot\left[\frac{1}{2}e^{-(t+4)} - e^{-2(t+4)} + \frac{1}{2}e^{-3(t+4)}\right]u(t+4)}$$

Problem 2 (10 points)

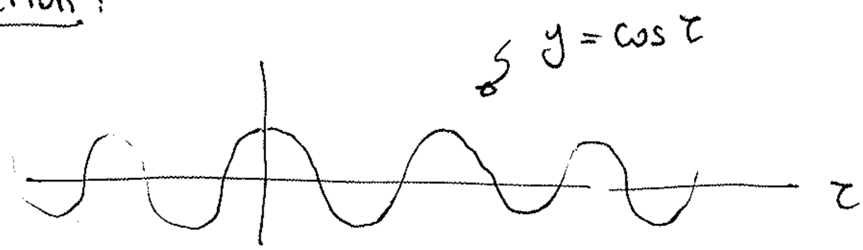
(a) Find $h(t) = (\text{rect} * \cos)(t)$. **[Hint: Calculate the convolution directly.]**

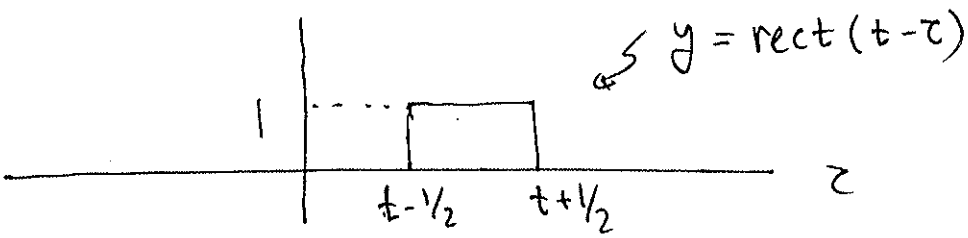
(b) Find the Fourier transform of $f(t) = e^{-|t|}$. **[Show your work.]**

(c) Use part (b) to compute the Fourier transform of $f_{\text{AM}}(t) = e^{-|t|}\cos(t)$.

Solution:

(a)





Then: $h(t) = \int_{t-1/2}^{t+1/2} 1 \cdot \cos\tau \, d\tau = \sin\tau \Big|_{t-1/2}^{t+1/2}$

$$\Rightarrow \boxed{h(t) = \sin(t+1/2) - \sin(t-1/2)}$$

(b) $\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-|t|} \cdot e^{-i\omega t}\, dt = \underbrace{\int_{-\infty}^{0} e^{t} e^{-i\omega t}\, dt}_{I_1} + \underbrace{\int_{0}^{\infty} e^{-t} e^{-i\omega t}\, dt}_{I_2}.$

$I_2 = \dfrac{1}{i\omega + 1}$ (this is the F.T. of $e^{-t}u(t)$).

$$I_1 = \int_{-\infty}^{0} e^{t(1-i\omega)}\,dt = \lim_{A\to-\infty} \left.\frac{e^{t(1-i\omega)}}{1-i\omega}\right|_{A}^{0}$$

$$= \frac{1}{1-i\omega} \quad \left(\text{since } \lim_{A\to-\infty} \frac{e^{A(1-i\omega)}}{1-i\omega} = 0\right)$$

So: $$\hat{f}(\omega) = I_1 + I_2 = \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{2}{1+\omega^2}$$

$$\Rightarrow \boxed{\hat{f}(\omega) = \frac{2}{1+\omega^2}}$$

(c) Recall that (i) $f\cdot g \xleftrightarrow{\text{F.T.}} (\hat{f} * \hat{g})\cdot\frac{1}{2\pi}$

(ii) $\cos t \longleftrightarrow \pi\left[\delta(\omega+1)+\delta(\omega-1)\right]$.

Then, ~~setting~~

$$\hat{f}_{AM}(\omega) = \left[\hat{f} * \pi\left[\delta(\omega+1)+\delta(\omega-1)\right]\right]\cdot\frac{1}{2\pi}$$

$$= \frac{1}{2}\hat{f}(\omega+1) + \frac{1}{2}\hat{f}(\omega-1)$$

$$= \frac{1}{2}\,\frac{2}{1+(\omega+1)^2} + \frac{1}{2}\cdot\frac{2}{1+(\omega-1)^2}$$

$$\boxed{\hat{f}_{AM}(\omega) = \frac{1}{1+(\omega+1)^2} + \frac{1}{1+(\omega-1)^2}}$$

Problem 3 (10 points)

Find the inverse Fourier transform of the following functions.

(a) $\widehat{f}(\omega) = \dfrac{1}{(2+i\omega)(5+i\omega)(6+i\omega)}$,

(b) $\widehat{g}(\omega) = \cos(5\omega + \pi/2)$.

Partial fraction exp.

(a) $$\frac{1}{(2+i\omega)(5+i\omega)(6+i\omega)} = \frac{1}{12}\frac{1}{2+i\omega} - \frac{1}{3}\cdot\frac{1}{5+i\omega} + \frac{1}{4}\frac{1}{6+i\omega}$$

$$\Rightarrow \boxed{f(t) = \left[\frac{1}{12}e^{-2t} - \frac{1}{3}e^{-5t} + \frac{1}{4}e^{-6t}\right]u(t)}$$

(b) (Very similar to a homework question):

$$\hat{g}(\omega) = \cos(5\omega + \pi/2) = \frac{e^{i(5\omega+\pi/2)} + e^{-i(5\omega+\pi/2)}}{2}$$

$$= \frac{1}{2}e^{i\pi/2}\cdot e^{i5\omega} + \frac{1}{2}e^{-i\pi/2}\cdot e^{-i5\omega}$$

$$\Rightarrow \hat{g}(\omega) = \frac{1}{2}i\,e^{i5\omega} - \frac{1}{2}i\,e^{-i5\omega}$$

Using $\delta(t+a) \longleftrightarrow e^{ia\omega}$, we then get

$$\boxed{g(t) = \frac{1}{2}i\,\delta(t+5) - \frac{1}{2}i\,\delta(t-5)}$$

Note: This question can also be solved using several properties of Fourier transform, such as time-shifting, scaling, and duality. If solved this way, one gets

$$\boxed{g(t) = \frac{1}{2} e^{-i\frac{\pi}{10}t}\left[\delta(t+5) + \delta(t-5)\right]}$$

Note that $e^{-i\frac{\pi}{10}t} \cdot \delta(t+5) = e^{-i\frac{\pi}{10}(-5)} \delta(t+5) = \frac{1}{2} i \delta(t+5)$

and $e^{-i\frac{\pi}{10}t} \delta(t-5) = e^{-i\frac{\pi}{10}\cdot 5} \delta(t-5) = -\frac{1}{2} i \delta(t-5)$

that is, both solutions agree.

Problem 4 (10 points)

Use properties of Fourier transforms to calculate the following integrals. **Show your work.**

(a) $\int_{-\infty}^{\infty} \frac{e^{i2\omega}}{5+iw} d\omega, = I_1$

(b) $\int_{-\infty}^{\infty} [\text{sinc}(\omega)]^2 d\omega, = I_2$

(c) $\int_{-\infty}^{\infty} [\text{sinc}(\omega)]^2 e^{i10\omega} d\omega. = I_3$

(a) Set $\hat{f}(\omega) = \frac{1}{5+i\omega}$. Then

using the inversion formula

$$I_1 = \int \frac{e^{i2\omega}}{5+i\omega} d\omega = \int \hat{f}(\omega) e^{i2\omega} d\omega = 2\pi f(2)$$

Since we know $f(t) = e^{-5t} u(t)$, we conclude

$$\boxed{I_1 = 2\pi f(2) = 2\pi \cdot e^{-10}}$$

(b) Here we will use Parseval's identity. Recall that

$\text{rect}(t) \longleftrightarrow \text{sinc}(\omega/2)$ which implies

$\frac{1}{2}\text{rect}\left(\frac{t}{2}\right) \longleftrightarrow \text{sinc}(\omega)$ (time-scaling).

So: $$\int_{-\infty}^{\infty} \left|\frac{1}{2}\text{rect}\left(\frac{t}{2}\right)\right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\text{sinc}(\omega)|^2 d\omega$$

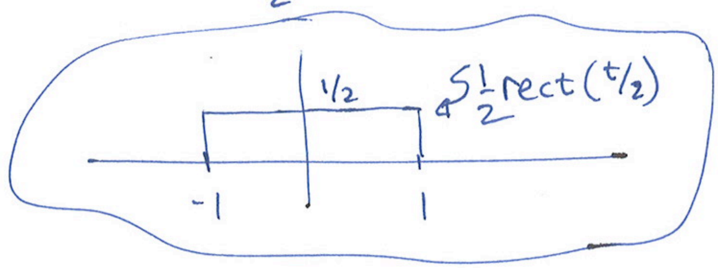
Parseval

For (b), the same method as (c) works. You do not need to use Parseval

So, $$I_2 = 2\pi \int_{-\infty}^{\infty} \left[\frac{1}{2} \operatorname{rect}\left(\frac{t}{2}\right) \right]^2 dt$$

$$= 2\pi \int_{-1}^{1} \frac{1}{4}\, dt = 2\pi \cdot \frac{1}{2} = \pi$$

$\leadsto \boxed{I_2 = \pi}$



(c) From (b), we know

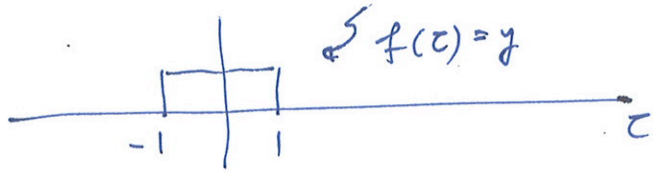
$$f(t) = \frac{1}{2}\operatorname{rect}\left(\frac{t}{2}\right) \longleftrightarrow \operatorname{sinc}(\omega) = \hat{f}(\omega)$$

Then using convolution thm, we know that

$$h(t) := (f * f)(t) \longleftrightarrow [\operatorname{sinc}(\omega)]^2 = \hat{h}(\omega)$$ ~~and so~~

$$I_3 = \int_{-\infty}^{\infty} \hat{h}(\omega) e^{i10\omega}\, d\omega = 2\pi \cdot h(10)$$

But $h = f * f$. So, $h(10) = \int f(\tau) f(10-\tau)\, d\tau$



$\Rightarrow f(\tau) \cdot f(10-\tau) = 0 \Rightarrow h(10) = 0 \Rightarrow \boxed{I_3 = 0}$