

Do the following SIX problems. Hand in written solutions for grading at the BEGINNING of the lecture on the due date. Illegible, disorganized or partial solutions will receive no credit. Staple your HW.

1. In each of the following, find all real  $\lambda$  such that the corresponding problem has nontrivial (i.e. not identically zero) solutions.

(a)  $y''(t) + \lambda y(t) = 0, \quad y(0) = 0, \quad y(1) = 0.$

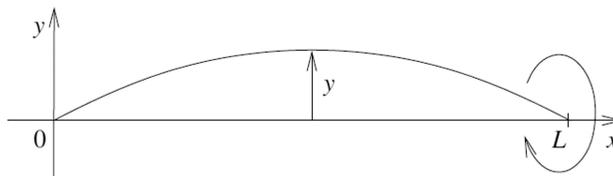
(b)  $y''(t) + \lambda y(t) = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$

2. In each of the following, find all real  $\lambda$  such that the corresponding problem has nontrivial (i.e. not identically zero) solutions.

(a)  $y''(t) + \lambda y(t) = 0, \quad y'(0) = 0, \quad y'(\pi) = 0.$

(b)  $y''(t) + \lambda y(t) = 0, \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi).$

3. Suppose that we have a tightly stretched elastic string of length  $L$  and uniform linear density  $\rho$ . Suppose that the string is **fixed at the beginning and end points**, and that the string quickly rotates at an angular velocity  $\omega$ , as the figure shows:



If we assume that the deflection  $y(x)$  is small, then we can use Newton's second law to get an equation

$$Ty''(x) + \rho\omega^2 y(x) = 0,$$

where  $T$  is the magnitude of string tension assumed to be constant everywhere. Assume that  $\rho$ ,  $T$  and  $L$  are fixed and we are changing  $\omega$ :

- (a) Find an expression for  $\omega$  in terms of  $\rho$ ,  $T$  and  $L$  such that the string is NOT in the equilibrium position (the equilibrium position is  $y(x) = 0$  for all  $x$ ).
- (b) For  $L = 3$  and  $T/\rho = 9$ , determine three values of  $\omega$  such that the string is NOT in the equilibrium position. **Sketch** the shape of the string for these three values of  $\omega$ .
4. Consider the wave equation with boundary conditions (here, we do not specify the initial conditions):

$$\begin{cases} u_{tt}(x, t) = u_{xx}(x, t), & \text{for all } 0 < x < L \text{ and } t > 0; \\ u_x(0, t) = 0, u_x(L, t) = 0, & \text{for all } t > 0. \end{cases}$$

(Notice the derivative in the boundary conditions!)

Find all nontrivial solutions  $u(x, t)$  of the following type:

$$u(x, t) = X(x)T(t)$$

where  $X(x)$  and  $T(t)$  are functions of the variables  $x$ ,  $t$ , respectively.

5. The temperature variation of a rod of length  $L$  is described by the heat equation

$$u_t(x, t) = \alpha^2 u_{xx}(x, t) \quad \text{for all } 0 < x < L \text{ and } t > 0.$$

Here  $u(x, t)$  is the temperature at position  $x$  and time  $t$ , and  $\alpha > 0$  is a constant depending on the material of the rod, called *thermal diffusivity*. (See the lecture notes "Derivation of the Heat Equation 1D" in the course website to see how this equation is derived.) Assume the boundary conditions:

$$u_x(0, t) = 0, \quad u_x(L, t) = 0, \quad \text{for all } t > 0.$$

(Notice the derivative in the boundary conditions!)

With this boundary condition, find all the solutions  $u(x, t)$  of the following type:

$$u(x, t) = X(x)T(t)$$

where  $X(x)$  and  $T(t)$  are functions of the variables  $x, t$ , respectively. (You may want to look at the notes "Solution of the Heat Equation by Separation of Variables" in the course website.)

6. Fix the constant  $\alpha^2 > 0$ . Consider the heat equation

$$u_t(x, t) = \alpha^2 u_{xx}(x, t), \quad \text{for all } 0 < x < L \text{ and } t > 0.$$

- (a) Subject to the boundary conditions  $u(0, t) = 0$  and  $u(L, t) = 0$  for all  $t > 0$ , solve the initial value problem if the temperature is initially  $u(x, 0) = 2 \sin(3\pi x/L)$ .
- (b) Subject to the boundary conditions  $u_x(0, t) = 0$  and  $u_x(L, t) = 0$  for all  $t > 0$ , solve the initial value problem if the temperature is initially  $u(x, 0) = 6 + 4 \cos(3\pi x/L)$ .