– Assignment #2 – Solutions –

#1 a) $\begin{cases} y''(t) + \lambda y(t) = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$

This BVP is exactly the same as the last example in Lecture 2 posted on the course website, with $L = 1$. (check details there).

This BVP has non zero solution when

$$\lambda = (n\pi)^2, \quad n = 1, 2, 3, \ldots$$

only positive integers are sufficient as negative integers give the same values for λ.

b) $\begin{cases} y''(t) + \lambda y(t) = 0 \\ y(0) = 0 \\ y'(\pi) = 0 \end{cases}$

- case $\lambda = 0$: $y'' = 0 \Rightarrow y(t) = C_2 + C_1 t$

$\Rightarrow y(0) = C_2 = 0$ and $y'(t) = C_1$

$\Rightarrow y'(\pi) = C_1 = 0 \Rightarrow y(t) = 0$

- <u>case $\lambda < 0$</u> : characteristic eq'n: $r^2 + \lambda = 0$

$$\Rightarrow r^2 = -\lambda = |\lambda| \Rightarrow r = \pm\sqrt{|\lambda|}$$

$$\Rightarrow y(t) = C_1 e^{\sqrt{|\lambda|}\,t} + C_2 e^{-\sqrt{|\lambda|}\,t}$$

and $y'(t) = \sqrt{|\lambda|}\left(C_1 e^{\sqrt{|\lambda|}\,t} - C_2 e^{-\sqrt{|\lambda|}\,t}\right)$

therefore, $y(0) = C_1 + C_2 = 0$

and $y'(\pi) = \sqrt{|\lambda|}\left(C_1 e^{\sqrt{|\lambda|}\,\pi} - C_2 e^{-\sqrt{|\lambda|}\,\pi}\right) = 0$

$\Rightarrow C_2 = -C_1$ and so,

$$\sqrt{|\lambda|}\; C_1 \left(e^{\sqrt{|\lambda|}\,\pi} + e^{-\sqrt{|\lambda|}\,\pi}\right) = 0$$

but the expression in parenthesis is never zero (why? check it!)

$\Rightarrow C_1 = 0 \Rightarrow C_2 = 0$ and so $y(t) = 0$,

- <u>case $\lambda > 0$</u> : $r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{\lambda}\, i$

$$\Rightarrow y(t) = C_1 \cos(\sqrt{\lambda}\, t) + C_2 \sin(\sqrt{\lambda}\, t)$$

and $y'(t) = -\sqrt{\lambda}\, C_1 \sin(\sqrt{\lambda}\, t) + \sqrt{\lambda}\, C_2 \cos(\sqrt{\lambda}\, t)$

Therefore,

$$y(0) = C_1 = 0 \quad \text{and so}$$

$$y'(\pi) = \sqrt{\lambda}\, C_2 \cos(\sqrt{\lambda}\,\pi) = 0$$

$$\Rightarrow \ (\text{for } C_2 \neq 0) \quad \cos(\sqrt{\lambda}\,\pi) = 0$$

$$\Rightarrow \quad \sqrt{\lambda}\,\pi = \frac{\pi}{2} + n\pi = \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow \quad \lambda = \left(n + \frac{1}{2}\right)^2, \quad n = 0, 1, 2, 3, \ldots$$

and so, $\quad y(t) = C_2 \sin\left(\left(n + \frac{1}{2}\right)t\right), \quad n = 0, 1, 2, 3, \ldots$

In summary: the BVP has nonzero solutions when $\lambda = \left(n + \frac{1}{2}\right)^2, \ n = 0, 1, 2, \ldots$

#2 a) $\begin{cases} y''(t) + \lambda y(t) = 0 \\ y'(0) = 0 \\ y'(\pi) = 0 \end{cases}$

case $\lambda = 0$: $y'' = 0 \Rightarrow y' = C_1 \Rightarrow y(t) = C_2 + C_1 t$

$$\Rightarrow y'(0) = C_1 = 0 \quad \text{and} \quad y'(\pi) = C_1 = 0$$

$$\Rightarrow \quad y(t) = C_2 \quad \text{for } \lambda = 0 \quad (\text{nonzero solutions})$$

case $\lambda < 0$: $r^2 + \lambda = 0 \Rightarrow r^2 = -\lambda = |\lambda| \Rightarrow r = \pm\sqrt{|\lambda|}$

$$y(t) = C_1 e^{\sqrt{|\lambda|}t} + C_2 e^{-\sqrt{|\lambda|}t}$$

$$\Rightarrow y'(t) = \sqrt{|\lambda|}\, C_1 e^{\sqrt{|\lambda|}t} - \sqrt{|\lambda|}\, C_2 e^{-\sqrt{|\lambda|}t}$$

$$\Rightarrow y'(0) = \sqrt{|\lambda|}\,(C_1 - C_2) = 0$$

and $y'(\pi) = \sqrt{|\lambda|}\,(C_1 e^{\sqrt{|\lambda|}\pi} - C_2 e^{-\sqrt{|\lambda|}\pi}) = 0$

$\Rightarrow C_1 = C_2$ and so

$$\sqrt{|\lambda|}\, C_1 (e^{\sqrt{|\lambda|}\pi} - e^{-\sqrt{|\lambda|}\pi}) = 0$$

$$\Rightarrow 2\sqrt{|\lambda|}\, C_1 \sinh(\sqrt{|\lambda|}\pi) = 0$$

but $\sinh(\sqrt{\lambda}\pi)$ is never zero for $\lambda \neq 0$

$\Rightarrow C_1 = 0$ and so $C_2 = 0 \Rightarrow y(t) = 0$

case $\lambda > 0$: $r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{\lambda}\, i$

$$\Rightarrow y(t) = C_1 \cos(\sqrt{\lambda}t) + C_2 \sin(\sqrt{\lambda}t)$$

$$\Rightarrow y'(t) = -\sqrt{\lambda}\, C_1 \sin(\sqrt{\lambda}t) + \sqrt{\lambda}\, C_2 \cos(\sqrt{\lambda}t)$$

$$\Rightarrow y'(0) = \sqrt{\lambda}\, C_2 = 0 \Rightarrow C_2 = 0$$

and $y'(\pi) = -\sqrt{\lambda}\, C_1 \sin(\sqrt{\lambda}\pi) = 0$

$\Rightarrow$ (for $C_1 \neq 0$) $\sin(\sqrt{\lambda}\pi) = 0$

$\Rightarrow \sqrt{\lambda}\pi = n\pi \Rightarrow \lambda = n^2, \quad n = 1, 2, 3, \ldots$

and so, $y(t) = C_1 \cos(nt), \quad n = 1, 2, 3, \ldots$

In summary: the BVP has nonzero solutions when $\lambda = n^2, \quad n = 0, 1, 2, 3, \ldots$

(this includes the nonzero solution when $\lambda = 0$)

b)
$$\begin{cases} y''(t) + \lambda y(t) = 0 \\ y(-\pi) = y(\pi) \\ y'(-\pi) = y'(\pi) \end{cases}$$

notice that the values or the derivatives of $y(t)$ are <u>NOT</u> specified at the endpoints, but rather they are the same at the beginning and at the end of the interval $[-\pi, \pi]$

<u>case $\lambda = 0$</u>:

$y(t) = C_2 + C_1 t \quad \Rightarrow \quad y'(t) = C_1$

so, $y(-\pi) = C_2 - C_1\pi$ and $y(\pi) = C_2 + C_1\pi$

and so $y(-\pi) = y(\pi) \Rightarrow 2C_1\pi = 0 \Rightarrow C_1 = 0$

and $y'(-\pi) = C_1 = y'(\pi)$ with $C_1 = 0$

$\Rightarrow$ the boundary conditions are redundant (they say nothing about C_2)

$\Rightarrow y(t) = C_2$ for $\lambda = 0$ (nonzero solutions)

case $\lambda < 0$:

$$y(t) = C_1 e^{\sqrt{|\lambda|}\,t} + C_2 e^{-\sqrt{|\lambda|}\,t}$$

and $y'(t) = \sqrt{|\lambda|}\,C_1 e^{\sqrt{|\lambda|}\,t} - \sqrt{|\lambda|}\,C_2 e^{-\sqrt{|\lambda|}\,t}$

Let's evaluate the boundary conditions,

$$y(-\pi) = C_1 e^{-\sqrt{|\lambda|}\,\pi} + C_2 e^{\sqrt{|\lambda|}\,\pi}$$

and $y(\pi) = C_1 e^{\sqrt{|\lambda|}\,\pi} + C_2 e^{-\sqrt{|\lambda|}\,\pi}$

and so, $y(-\pi) = y(\pi)$

$$\Rightarrow C_1 e^{-\sqrt{|\lambda|}\,\pi} + C_2 e^{\sqrt{|\lambda|}\,\pi} = C_1 e^{\sqrt{|\lambda|}\,\pi} + C_2 e^{-\sqrt{|\lambda|}\,\pi}$$

$$\Rightarrow C_2\left(e^{\sqrt{|\lambda|}\,\pi} - e^{-\sqrt{|\lambda|}\,\pi}\right) = C_1\left(e^{\sqrt{|\lambda|}\,\pi} - e^{-\sqrt{|\lambda|}\,\pi}\right)$$

$\Rightarrow C_1 = C_2$ since the expression in parenthesis is <u>NOT</u> zero (we have seen this before, check it!)

Also, $y'(-\pi) = y'(\pi)$

$$\Rightarrow \sqrt{|\lambda|}\, c_1 e^{-\sqrt{|\lambda|}\pi} - \sqrt{|\lambda|}\, c_2 e^{\sqrt{|\lambda|}\pi} = \sqrt{|\lambda|}\, c_1 e^{\sqrt{|\lambda|}\pi} - \sqrt{|\lambda|}\, c_2 e^{-\sqrt{|\lambda|}\pi}$$

$\Rightarrow$ (since $c_1 = c_2$)

$$-\sqrt{|\lambda|}\, c_1 \left(e^{\sqrt{|\lambda|}\pi} - e^{-\sqrt{|\lambda|}\pi}\right) = \sqrt{|\lambda|}\, c_1 \left(e^{\sqrt{|\lambda|}\pi} - e^{-\sqrt{|\lambda|}\pi}\right)$$

$\Rightarrow -c_1 = c_1$ since $\lambda \neq 0$ and the expression in parenthesis is different from zero as well

$\Rightarrow c_1 = 0 \Rightarrow c_2 = 0 \Rightarrow y(t) = 0$

<u>case $\lambda > 0$:</u>

$$y(t) = c_1 \cos(\sqrt{\lambda}\, t) + c_2 \sin(\sqrt{\lambda}\, t)$$

and $y'(t) = -\sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda}\, t) + \sqrt{\lambda}\, c_2 \cos(\sqrt{\lambda}\, t)$

evaluate the boundary conditions:

$$y(-\pi) = c_1 \cos(-\sqrt{\lambda}\,\pi) + c_2 \sin(-\sqrt{\lambda}\,\pi)$$
$$= c_1 \cos(\sqrt{\lambda}\,\pi) - c_2 \sin(\sqrt{\lambda}\,\pi)$$

and $y(\pi) = c_1 \cos(\sqrt{\lambda}\,\pi) + c_2 \sin(\sqrt{\lambda}\,\pi)$, and so $y(-\pi) = y(\pi)$

$\Rightarrow c_1 \cos(\sqrt{\lambda}\pi) - c_2 \sin(\sqrt{\lambda}\pi) = c_1 \cos(\sqrt{\lambda}\pi) + c_2 \sin(\sqrt{\lambda}\pi)$

$\Rightarrow 2c_2 \sin(\sqrt{\lambda}\pi) = 0$

$\Rightarrow$ (for $c_2 \neq 0$) $\sin(\sqrt{\lambda}\pi) = 0$

$\Rightarrow \sqrt{\lambda}\pi = n\pi \Rightarrow \lambda = n^2$, $n = 1, 2, 3, \ldots$

Also, $y'(-\pi) = -\sqrt{\lambda} c_1 \sin(-\sqrt{\lambda}\pi) + \sqrt{\lambda} c_2 \cos(-\sqrt{\lambda}\pi)$
$= \sqrt{\lambda} c_1 \sin(\sqrt{\lambda}\pi) + \sqrt{\lambda} c_2 \cos(\sqrt{\lambda}\pi)$

and $y'(\pi) = -\sqrt{\lambda} c_1 \sin(\sqrt{\lambda}\pi) + \sqrt{\lambda} c_2 \cos(\sqrt{\lambda}\pi)$,

and so, $y'(-\pi) = y'(\pi)$

$\Rightarrow \sqrt{\lambda} c_1 \sin(\sqrt{\lambda}\pi) + \sqrt{\lambda} c_2 \cos(\sqrt{\lambda}\pi) = -\sqrt{\lambda} c_1 \sin(\sqrt{\lambda}\pi) + \sqrt{\lambda} c_2 \cos(\sqrt{\lambda}\pi)$

$\Rightarrow 2\sqrt{\lambda} c_1 \sin(\sqrt{\lambda}\pi) = 0$

$\Rightarrow$ (for $c_1 \neq 0$) $\sin(\sqrt{\lambda}\pi) = 0$ (as above)

$\Rightarrow \lambda = n^2$, $n = 1, 2, 3,$

Since for $\lambda = n^2$ we have $c_1 \neq 0$ and $c_2 \neq 0$, then

$y(t) = c_1 \cos(nt) + c_2 \sin(nt)$ (non zero solutions)

In summary: the BVP has nonzero solutions

when $\lambda = n^2$, $n = 0, 1, 2, 3, \ldots$

(this includes the nonzero solution when $\lambda = 0$)

#3 We are given the BVP,

$$\begin{cases} y''(x) + \dfrac{\rho w^2}{T} y(x) = 0 \\ y(0) = 0 \\ y(L) = 0 \end{cases}$$

a) We look for nonzero solutions (the equilibrium position $y(x) = 0$ is the zero solution!)

This is the same type of BVP discussed in Lecture 2 (check details there), with

$$\frac{\rho w^2}{T} = \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$ for nonzero solutions

$$\Rightarrow \quad w^2 = \left(\frac{n\pi}{L}\right)^2 \frac{T}{\rho}$$

Therefore, $\omega = \dfrac{n\pi}{L}\sqrt{\dfrac{T}{\rho}}$, $n = 1, 2, 3, \ldots$

b) For $L=3$ and $\frac{T}{\varphi}=9$ we have

$$\omega = \frac{n\pi}{3}\sqrt{9} = n\pi, \quad n=1,2,3,\ldots$$

three values of ω such that y is nonzero are

$$\omega=\pi, \quad \omega=2\pi \text{ and } \omega=3\pi$$

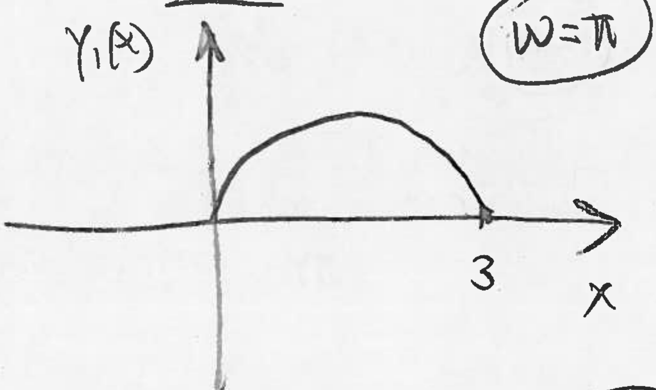
with solutions (Recall: $y(x) = C\sin\left(\frac{n\pi}{L}x\right)$)

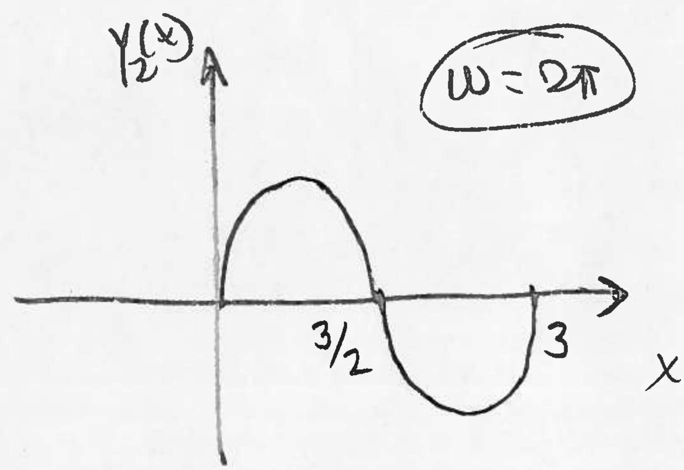
$y_1(x) = \sin\left(\frac{\pi}{3}x\right)$ when $n=1$ and so $\omega=\pi$

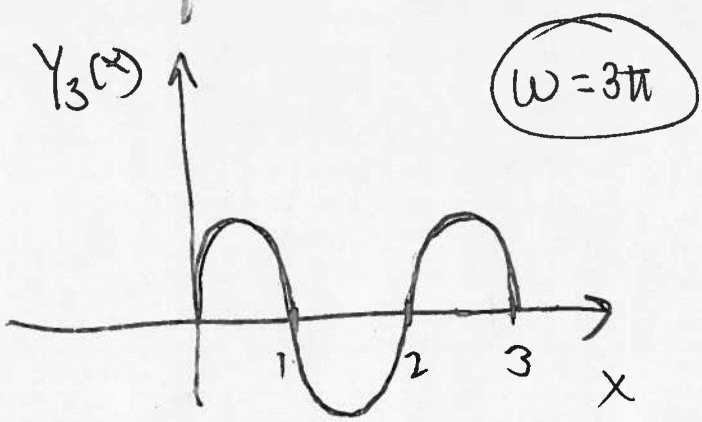
$y_2(x) = \sin\left(\frac{2\pi}{3}x\right)$ when $n=2$ and so $\omega=2\pi$

$y_3(x) = \sin(\pi x)$ when $n=3$ and so $\omega=3\pi$

Sketch:







Notice that we set $C=1$ because the actual value does NOT affect the shape of the string

#4
$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < L \text{ and } t > 0 \quad \text{(PDE)} \\ u_x(0,t) = u_x(L,t) = 0, & t > 0 \quad \text{(BC)} \end{cases}$$

Step 1: Hypothesize $u(x,t) = X(x)T(t)$

$\Rightarrow$ $u_{tt}(x,t) = X(x)T''(t)$

and $u_{xx}(x,t) = X''(x)T(t)$

$\Rightarrow$ $X(x)T''(t) = X''(x)T(t)$

$\Rightarrow$ $\dfrac{T''(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$

$\Rightarrow$ $X''(x) + \lambda X(x) = 0$

and $T''(t) + \lambda T(t) = 0$

Step 2 Recall: $u_x(0,t) = 0$ and $u_x(L,t) = 0$

$\Rightarrow$ $u_x(x,t) = X'(x)T(t)$ and so,

$u_x(0,t) = X'(0)T(t) = 0$ for all $t > 0$

$u_x(L,t) = X'(L)T(t) = 0$ for all $t > 0$

$\Rightarrow$ $X'(0) = 0$ and $X'(L) = 0$

Then, we have the following BVP

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = 0 \\ X'(L) = 0 \end{cases}$$

This BVP is similar to the one in Problem #2 (a), only there $L = \pi$. Thus, we can reason similarly to find that for nonzero solutions:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, 3, \ldots \quad \text{(eigenvalues)}$$

and $X_n(x) = \cos\left(\frac{n\pi}{L}x\right)$ (eigenfunctions)

(Notice that we have included the eigenvalue $\lambda = 0$ with eigenfunction $X_0 = 1$)

On the other hand,

$$T''(t) + \left(\frac{n\pi}{L}\right)^2 T(t) = 0, \quad n = 0, 1, 2, \ldots$$

$\underline{n=0}$: $T''(t) = 0 \Rightarrow T(t) = B_0 + A_0 t$

$\underline{n>0}$: $T_n(t) = A_n \cos\left(\frac{n\pi}{L}t\right) + B_n \sin\left(\frac{n\pi}{L}t\right)$

Thus, all nontrivial solutions of the type

$$u(x,t) = X(x)T(t) \text{ are}$$

$$u_0(x,t) = X_0(x)T_0(t) = A_0 t + B_0,$$

$$u_n(x,t) = X_n(x)T_n(t)$$

$$= \cos\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{n\pi}{L}t\right) + B_n \sin\left(\frac{n\pi}{L}t\right)\right]$$

$$n = 1, 2, 3, \ldots$$

#5 This problem is similar to Problem #4, only we are dealing with a heat equation

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u_x(0,t) = u_x(L,t) = 0, \quad t > 0 \end{cases}$$

we use separation of variables,

Step 1 Hypothesize $u(x,t) = X(x)T(t)$

$$\Rightarrow X(x)T'(t) = \alpha^2 X''(x)T(t)$$

$$\Rightarrow \frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$\Rightarrow \quad X''(x) + \lambda X(x) = 0$

and $\quad T'(t) + \lambda \alpha^2 T(t) = 0$

Step 2: Recall: $u_x(0,t) = u_x(L,t) = 0$

$$\Rightarrow \quad X'(0) = X'(L) = 0$$

and so, we have the BVP

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X'(L) = 0 \end{cases}$$

with nonzero solution when (see Problem #4)

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, 3, \ldots$$

and so $\quad X_n(x) = \cos\left(\frac{n\pi}{L} x\right)$

(this includes the eigenvalue $\lambda = 0$)
with eigenfunction $X_0 = 1$

On the other hand,

$$T'(t) + \left(\frac{n\pi\alpha}{L}\right)^2 T(t) = 0, \quad n = 0, 1, 2, \ldots$$

$\underline{n=0}$: $\quad T'(t) = 0 \quad \Rightarrow \quad T_0(t) = A_0$

$\underline{n>0}$: $\quad T_n(t) = A_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$

Thus, all nontrivial solutions of the type $u(x,t) = X(x)T(t)$ are

$$u_n(x,t) = A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}, \quad n=0,1,2,\ldots$$

(notice that this includes the case $n=0$)

#6 a) $$\begin{cases} u_t = \alpha^2 u_{xx}, \; 0<x<L, \; t>0 & \text{(PDE)} \\ u(0,t) = u(L,t) = 0 \quad t>0 & \text{(BC)} \\ u(x,0) = 2\sin\left(\frac{3\pi x}{L}\right) \; 0<x<L & \text{(IC)} \end{cases}$$

we use the method of separation of variables, well-known by now. Thus,

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$\Rightarrow \quad X''(x) + \lambda X(x) = 0$$
$$T'(t) + \lambda\alpha^2 T(t) = 0$$

(BC) $\Rightarrow$ $X(0) = X(L) = 0$

$$\Rightarrow \quad \text{BVP} \begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X(L) = 0 \end{cases}$$

with nonzero solution when (see example in Lecture 2)

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots \quad \text{(eigenvalues)}$$

and so, $X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$ (eigenfunctions)

Also, $T'(t) + \left(\frac{n\pi\alpha}{L}\right)^2 T(t) = 0$

$$\Rightarrow T(t) = A_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

$$\Rightarrow u_n(x,t) = X_n(x)\,T_n(t)$$

$$= A_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}, \quad n = 1, 2, 3, \ldots$$

and so

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$ by the superposition principle

(IC) $\Rightarrow$

$$u(x,0) = 2\sin\left(\frac{3\pi}{L}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

$\Rightarrow \quad 2\sin\left(\frac{3\pi}{L}\right) = A_1 \sin\left(\frac{\pi}{L}x\right) + A_2 \sin\left(\frac{2\pi}{L}x\right) + \cdots$

$\Rightarrow \quad A_n \neq 0$ when $n = 3$ and $A_n = 0$ otherwise

$\Rightarrow \quad 2\sin\left(\frac{3\pi}{L}\right) = A_3 \sin\left(\frac{3\pi}{L}\right)$

$\Rightarrow \quad A_3 = 2$

finally, $\quad u(x,t) = 2\sin\left(\frac{3\pi}{L}x\right) e^{-\left(\frac{3\pi\alpha}{L}\right)^2 t}$

b) $\begin{cases} u_t = \alpha^2 u_{xx}, & 0 < x < L,\ t > 0 \quad \text{(PDE)} \\ u_x(0,t) = u_x(L,t) = 0, & t > 0 \quad \text{(BC)} \\ u(x,0) = 6 + 4\cos\left(\frac{3\pi x}{L}\right), & 0 < x < L \quad \text{(IC)} \end{cases}$

As in Problem #5, we find that

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

$\text{(IC)} \Rightarrow u(x,0) = 6 + 4\cos\left(\frac{3\pi x}{L}\right) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$

$\Rightarrow \quad 6 + 4\cos\left(\frac{3\pi x}{L}\right) = A_0 + A_1\cos\left(\frac{\pi}{L}x\right) + A_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$

$\Rightarrow \quad A_0 = 6,\ A_3 = 4$ and $A_n = 0$ otherwise

$$\Rightarrow u(x,t) = 6 + 4\cos\left(\frac{3\pi x}{L}\right) e^{-\left(\frac{3\pi\alpha}{L}\right)^2 t}$$