– Assignment #1 – Solutions –

(#1) $z = -2 + 2\sqrt{3}\,i$ and $w = 2i$

a) $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$r^2 = (-2)^2 + (2\sqrt{3})^2 = 16 \therefore r = 4$

$\Rightarrow z = 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$\Rightarrow \tan\theta = -\sqrt{3}$ with $\sin\theta = \frac{\sqrt{3}}{2} > 0$

$\cos\theta = -\frac{1}{2} < 0$

$\Rightarrow \theta = \frac{2\pi}{3}$ and so,

$z = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ polar form

$z = 4e^{\frac{2\pi}{3}i}$ exponential form

Similarly, for $w = 2i$ we have,

$w = 2\sin\frac{\pi}{2}\,i$ polar form

$w = 2e^{\frac{\pi}{2}i}$ exponential form

b) Then, we have

$$z^3 = 4^3 e^{2\pi i} = 64$$

$$\overline{w} = 2e^{-\frac{\pi}{2}i} \text{ and so } (\overline{w})^{-2} = \frac{1}{4} e^{\pi i} = -\frac{1}{4}$$

$$\Rightarrow z^3 + (\overline{w})^{-2} = 64 - \frac{1}{4} = \frac{255}{4} = 63\tfrac{3}{4}$$

#2 $z^3 + 27 = 0$,

a) Recall, $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\Rightarrow z^3 = r^3 e^{3\theta i} = -27 = -3^3$$

$\therefore r = 3$ and $e^{3\theta i} = \cos 3\theta + i\sin 3\theta = -1$

Thus, $\cos 3\theta = -1$ and $\sin 3\theta = 0$

$\Rightarrow 3\theta = \pi + 2k\pi = (2k+1)\pi$, k integer

$\Rightarrow \theta = (2k+1)\frac{\pi}{3}$, k integer

Recall, $-\pi < \theta \leq \pi$ and so

$$\begin{cases} k=-1, & \theta = -\frac{\pi}{3} \\ k=0, & \theta = \frac{\pi}{3} \\ k=1, & \theta = \pi \end{cases} \Rightarrow \begin{cases} z_1 = 3e^{-\frac{\pi}{3}i} \\ z_2 = 3e^{\frac{\pi}{3}i} \\ z_3 = 3e^{\pi i} \end{cases}$$

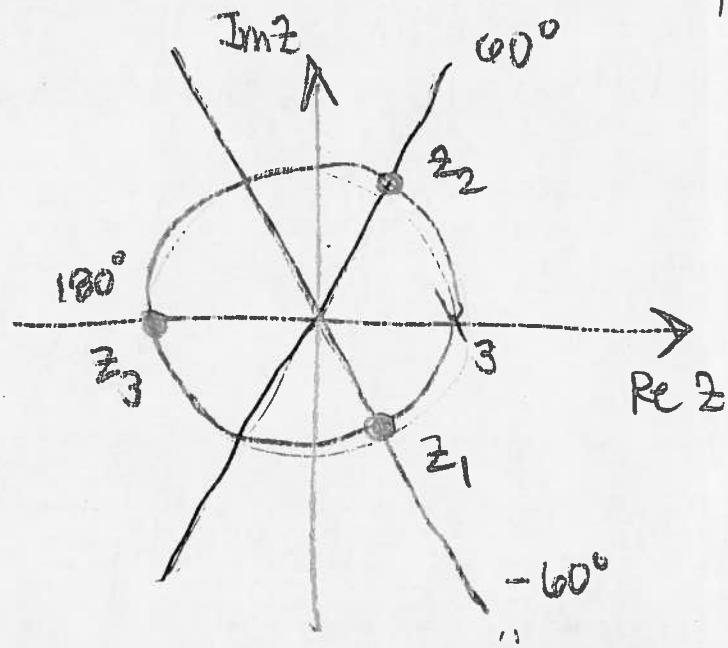
z_1, z_2 and z_3 are the three roots of $z^3 + 27 = 0$

Check:

$z_1^3 = 27e^{-\pi i} = -27$ ✓

$z_2^3 = 27e^{\pi i} = -27$ ✓

$z_3^3 = 27e^{3\pi i} = -27$ ✓



Roots on the complex plane

b) $f(t) = 2\sin 3t + 4\cos 5t$

Recall, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$$\therefore f(t) = 2\left(\frac{e^{3ti} - e^{-3ti}}{2i}\right) + 4\left(\frac{e^{5ti} + e^{-5ti}}{2}\right)$$

$$= -ie^{3ti} + ie^{-3ti} + 2e^{5ti} + 2e^{-5ti}$$

#3 We use the hint to find that,

$$\sum_{n=0}^{10} e^{\frac{2\pi n}{10}i} = \sum_{n=0}^{10}\left(e^{\frac{2\pi i}{10}}\right)^n = \frac{1-\left(e^{\frac{2\pi i}{10}}\right)^{10+1}}{1-e^{\frac{2\pi i}{10}}} = \frac{1-e^{2\pi i}e^{\frac{2\pi}{10}i}}{1-e^{\frac{2\pi i}{10}}}$$

by hint (formula for geometric sum with $r = e^{\frac{2\pi i}{10}}$)

but, $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$

$$\Rightarrow \sum_{n=0}^{10} e^{\frac{2\pi n}{10} i} = \frac{1 - e^{\frac{2\pi}{10} i}}{1 - e^{\frac{2\pi}{10} i}} = 1$$

#4 a) IVP $\begin{cases} 4y'' + 4y' + 17y = 0 & \text{(ODE)} \\ y(0) = 1 & \\ y'(0) = 2 & \text{(IC)} \end{cases}$

characteristic equation : $4r^2 + 4r + 17 = 0$

$$\Rightarrow r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 4 \cdot 17}}{2 \cdot 4} = \frac{-4 \pm \sqrt{-16^2}}{8} = -\frac{1}{2} \pm 2i$$

$$\Rightarrow r_1 = -\frac{1}{2} + 2i \text{ and } r_2 = -\frac{1}{2} - 2i$$

$$\Rightarrow y(t) = e^{-\frac{1}{2}t}\left[C_1 \cos(2t) + C_2 \sin(2t)\right]$$ general solution

Evaluate initial conditions :

$$y(0) = C_1 = 1$$

and $y'(t) = -\frac{1}{2} y(t) + e^{-\frac{1}{2}t}\left[-2C_1 \sin(2t) + 2C_2 \cos(2t)\right]$

$$\Rightarrow y'(0) = -\frac{1}{2} y(0) + 2C_2 = 2$$

$$\Rightarrow 2C_2 = 2 + \frac{1}{2} \cdot 1 = \frac{5}{2} \Rightarrow C_2 = \frac{5}{4}$$

$\therefore\ y(t) = e^{-\frac{1}{2}t}\left[\cos(2t) + \frac{5}{4}\sin(2t)\right]$ solution to the IVP ▪

sketch given at the end of the document.

b) IVP $\begin{cases} y'' + y = 2\cos t & \text{(ODE)} \\ y(0) = 1 & \text{(IC)} \\ y'(0) = 0 & \end{cases}$

▪ Homogeneous equation:

$y_H'' + y_H = 0$

characteristic eq'n : $r^2 + 1 = 0$

$\Rightarrow\ r = \pm i \ \Rightarrow\ r_1 = i$ and $r_2 = -i$

$\Rightarrow\ y_H(t) = C_1\cos t + C_2\sin t$

▪ Particular solution

we use the method of undetermined coefficients.

we start with an educated guess based on the function on the right-hand side of the ODE : $2\cos t$

first guess : $y_P = A\cos t + B\sin t$

but this guess does NOT work (show it!!)

(Notice that it has the form of $y_H(t)$!!)

<u>second guess</u>: $y_p(t) = At\cos t + Bt\sin t$

$\Rightarrow \quad y_p'(t) = A\cos t + B\sin t - At\sin t + Bt\cos t$

and $\quad y_p''(t) = -A\sin t + B\cos t - A\sin t + B\cos t$

$\qquad\qquad\qquad - At\cos t - Bt\sin t$

$\qquad = -2A\sin t + 2B\cos t - At\cos t - Bt\sin t$

$\Rightarrow \quad y_p''(t) + y_p(t) = -2A\sin t + 2B\cos t - \cancel{At\cos t}$

$\qquad\qquad\qquad - \cancel{Bt\sin t} + \cancel{At\cos t} + \cancel{Bt\sin t}$

$\qquad = -2A\sin t + 2B\cos t$

$\qquad = 2\cos t$ (because $y_p(t)$ <u>must</u> satisfy the ODE)

$\Rightarrow \quad -2A = 0$ and $2B = 2$

$\Rightarrow \quad A = 0$ and $B = 1$

$\therefore \quad y_p(t) = t\sin t$

<u>check</u>: $y_p'(t) = \sin t + t\cos t$

$y_p''(t) = \cos t + \cos t - t\sin t$

$\Rightarrow \quad y_p''(t) + y_p(t) = 2\cos t - t\sin t + t\sin t = 2\cos t$ ✓

$\therefore$ $y_p(t)$ satisfies the ODE

$\therefore\ y(t) = y_H(t) + y_p(t) = C_1\cos t + C_2\sin t + t\sin t$ general solution

- Evaluate initial conditions

$y(0) = C_1 = 1$

and $y'(t) = -C_1\sin t + C_2\cos t + \sin t + t\cos t$

$\Rightarrow y'(0) = C_2 = 0$

$\therefore\ y(t) = \cos t + t\sin t$ solution to the IVP

Sketch given at the end of the document

c) IVP $\begin{cases} y'' + 7y' = e^{-7t} & \text{(ODE)} \\ y(0) = 0 & \\ y'(0) = 0 & \text{(IC)} \end{cases}$

- Homogeneous equation:

$y_H'' + 7y_H' = 0$

characteristic eqt'n: $r^2 + 7r = 0$

$\Rightarrow r(r+7) = 0 \Rightarrow r_1 = 0$ and $r = -7$

$\Rightarrow y_H(t) = C_1 + C_2 e^{-7t}$

- Particular solution

again, we use the method of undetermined coefficients

first guess: $y_p(t) = Ae^{-7t}$ does NOT work (show it!!)

(Again, notice that it has the form of $y_H(t)$)

second guess: $y_p(t) = Ate^{-7t}$

$$\Rightarrow y'_p(t) = Ae^{-7t} - 7Ate^{-7t}$$

and

$$y''_p(t) = -7Ae^{-7t} + 49Ate^{-7t} - 7Ae^{-7t}$$

$$= -14Ae^{-7t} + 49Ate^{-7t}$$

$$\Rightarrow y''_p + 7y'_p = -14Ae^{-7t} + \cancel{49Ate^{-7t}} + 7Ae^{-7t} - \cancel{49Ate^{-7t}}$$

$$= -7Ae^{-7t}$$

$$= e^{-7t}$$ (because $y_p(t)$ must satisfy the ODE)

$$\Rightarrow -7A = 1 \Rightarrow A = -\frac{1}{7}$$

$$\therefore y_p(t) = -\frac{1}{7}te^{-7t}$$

Check: $y_p'(t) = -\frac{1}{7}e^{-7t} + te^{-7t}$

$y_p''(t) = e^{-7t} + e^{-7t} - 7te^{-7t}$

$$\Rightarrow y_p''(t) + 7y_p'(t) = 2e^{-7t} - \cancel{7te^{-7t}} - e^{-7t} + \cancel{7te^{-7t}}$$

$$= e^{-7t} \checkmark$$

$y_p(t)$ satisfies the ODE

$$\therefore\ y(t) = C_1 + C_2 e^{-7t} - \frac{1}{7}te^{-7t}$$ general solution

- Evaluate initial conditions

$$y(0) = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$$

and $y'(t) = -7C_2 e^{-7t} - \frac{1}{7}e^{-7t} + te^{-7t}$

$$\Rightarrow y'(0) = -7C_2 - \frac{1}{7} = 0 \Rightarrow C_2 = -\frac{1}{49}$$

$$\therefore\ C_1 = \frac{1}{49} \text{ and } C_2 = -\frac{1}{49}$$

$$\Rightarrow y(t) = \frac{1}{49} - \frac{1}{49}e^{-7t} - \frac{1}{7}te^{-7t}$$

or $y(t) = \frac{1}{49} - \frac{1}{49}(1+7t)e^{-7t}$ solution to the IVP

sketch given at the end of the document

#5

a) BVP $\begin{cases} y'' - 10y' + 25y = 0 & \text{(ODE)} \\ y(0) = 1 & \text{(BC)} \\ y(1) = 0 & \end{cases}$

characteristic equation: $r^2 - 10r + 25 = 0$

$\Rightarrow (r-5)^2 = 0 \Rightarrow r = 5$ with multiplicity 2

$\therefore \quad y(x) = (C_1 + C_2 x) e^{5x}$ general solution (obtained as discussed in class by using the reduction of order technique)

- Evaluate boundary conditions:

$y(0) = C_1 = 1$

and $y(1) = (C_1 + C_2) e^{5} = 0 \Rightarrow C_1 + C_2 = 0$

$\Rightarrow C_1 = 1$ and $C_2 = -1$

$\therefore \quad y(x) = (1-x) e^{5x}$ solution to the BVP

(It is a good idea to check that, indeed, $y(x)$ satisfies the ODE and the BC)

b) BVP $\begin{cases} y'' + 4y' + 7y = 0 & \text{(ODE)} \\ y(0) = 2 & \\ y\left(\frac{\sqrt{3}\pi}{6}\right) = 1 & \text{(B.C)} \end{cases}$

characteristic equation = $r^2 + 4r + 7 = 0$

$\Rightarrow r = \dfrac{-4 \pm \sqrt{4^2 - 4 \cdot 7}}{2} = \dfrac{-4 \pm \sqrt{-12}}{2} = -2 \pm \sqrt{3}\, i$

$\Rightarrow r_1 = -2 + \sqrt{3}\, i$ and $r_2 = -2 - \sqrt{3}\, i$

$\therefore y(x) = e^{-2x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right)$ general solution

. Evaluate boundary conditions:

$y(0) = C_1 = 2$

$y\left(\frac{\sqrt{3}\pi}{6}\right) = e^{-\frac{\sqrt{3}}{3}\pi}\left(C_1 \cos\frac{\pi}{2} + C_2 \sin\frac{\pi}{2}\right)$

$= C_2 e^{-\frac{\sqrt{3}}{3}\pi} = 1$

$\therefore C_1 = 2$ and $C_2 = e^{\frac{\sqrt{3}}{3}\pi}$

$\Rightarrow y(x) = e^{-2x}\left(2\cos\sqrt{3}x + e^{\frac{\sqrt{3}}{3}\pi} \sin\sqrt{3}x\right)$ solution to the BVP

#6

a) We are told that

$$y_1'' + P(x) y_1' + Q(x) y_1 = R_1(x) \qquad (1)$$

and $$y_2'' + P(x) y_2' + Q(x) y_2 = R_2(x) \qquad (2)$$

Let $y(x) = y_1(x) + y_2(x)$. Then,

$$y'' + P(x) y' + Q(x) y = (y_1 + y_2)'' + P(x)(y_1 + y_2)' + Q(x)(y_1 + y_2)$$

$$= y_1'' + y_2'' + P(x)(y_1' + y_2') + Q(x)(y_1 + y_2) \quad \text{by properties of derivatives}$$

$$= y_1'' + P(x) y_1' + Q(x) y_1 + y_2'' + P(x) y_2' + Q(x) y_2$$

$$= R_1(x) + R_2(x) \quad \text{by (1) and (2)}$$

b) We look for the general solution of:

$$y'' + 4y = 4\cos 2x + 6\cos x + 8x^2 - 4x$$

2nd order, linear, non-homogeneous ODE

* Homogeneous eqt'n.

$$y'' + 4y = 0$$

characteristic eqt'n: $r^2 + 4 = 0$

$\Rightarrow \quad r = \pm 2i$

$\therefore \quad y_H(x) = C_1 \cos(2x) + C_2 \sin(2x)$

- Particular solution

we look for particular solutions to the following eqtns.

$$y'' + 4y = 8x^2 - 4x \quad (1)$$

and $$y'' + 4y = 4\cos 2x + 6\cos x \quad (2)$$

then we use the principle of superposition in part (a)

Equation (1):

guess: $y_1(x) = Ax^2 + Bx + C$

$\Rightarrow y_1'(x) = 2Ax + B$ and $y_1''(x) = 2A$

$\Rightarrow y_1'' + 4y_1 = 2A + 4(Ax^2 + Bx + C) = 8x^2 - 4x$ by (1)

$\Rightarrow 4Ax^2 + 4Bx + 2A + 4C = 8x^2 - 4x$

$\Rightarrow 4A = 8$, $4B = -4$, $2A + 4C = 0$ | $\therefore A = 2$, $B = -1$, $4C = -2A \Rightarrow C = -1$

$\therefore \quad y_1(x) = 2x^2 - x - 1$

Check: $y_1'' + 4y_1 = 4 + 4(2x^2 - x - 1) = 8x^2 - 4x$ ✓

Equation (2):

guess: $Y_2(x) = Ax\cos 2x + Bx\sin 2x + C\cos x + D\sin x$

$\Rightarrow Y_2' = A\cos 2x + B\sin 2x - 2Ax\sin 2x + 2Bx\cos 2x$
$- C\sin x + D\cos x$

and $Y_2'' = -2A\sin 2x + 2B\cos 2x - 2A\sin 2x + 2B\cos 2x$
$- 4Ax\cos 2x - 4Bx\sin 2x - C\cos x - D\sin x$

$\therefore Y_2'' + 4Y_2 = -4A\sin 2x + 4B\cos 2x - \cancel{4Ax\cos 2x} - \cancel{4Bx\sin 2x}$
$- C\cos x - D\sin x + \cancel{4Ax\cos 2x} + \cancel{4Bx\sin 2x}$
$+ 4C\cos x + 4D\sin x$

$= -4A\sin 2x + 4B\cos 2x + 3C\cos x + 3D\sin x$

$= 4\cos 2x + 6\cos x$ by (2)

$\Rightarrow$ $-4A = 0$ $\quad 3C = 6$ $\quad$ | $\Rightarrow$ $A = 0$ $\quad C = 2$
$4B = 4$ $\quad 3D = 0$ $\quad$ | $B = 1$ $\quad D = 0$

$\therefore Y_2(x) = x\sin 2x + 2\cos x$

Check: $Y_2'' + 4Y_2 = 2\cos 2x + 2\cos 2x - \cancel{4x\sin 2x} - 2\cos x$
$+ \cancel{4x\sin 2x} + 8\cos x = 4\cos 2x + 6\cos x$ ✓

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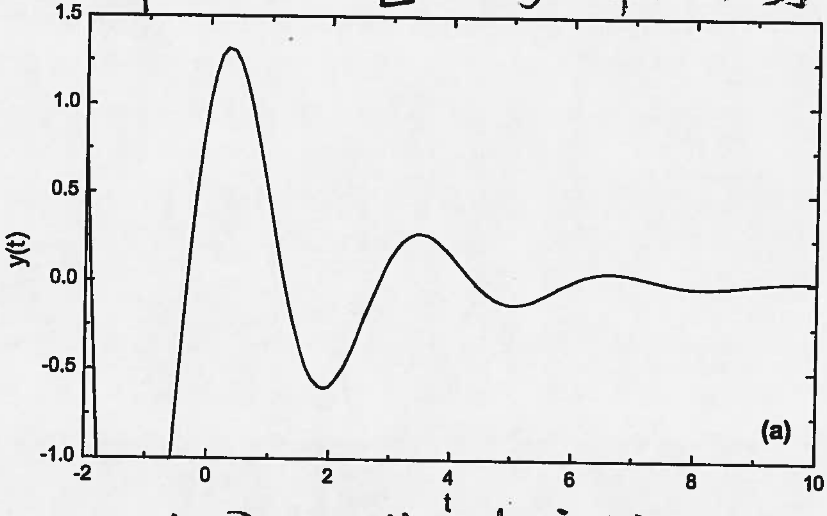
then, by the superposition principle:

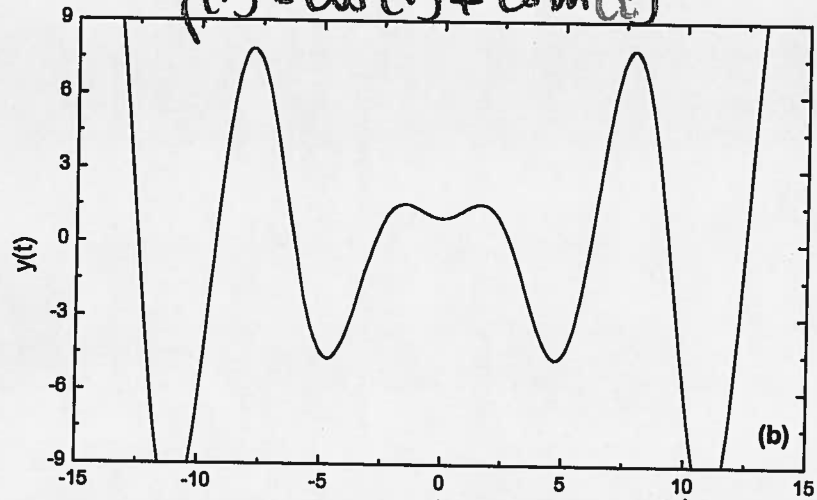
$Y(x) = C_1\cos(2x) + C_2\sin(2x) + 2x^2 - x - 1 + x\sin 2x + 2\cos x$ general solution.

- Sketches for Problem 3 -

$$y(t) = e^{-\frac{1}{2}t}\left[\cos(2t) + \frac{5}{4}\sin(2t)\right]$$



$$y(t) = \cos(t) + t\sin(t)$$



$$y(t) = \frac{1}{49} - \frac{1}{49}(1+7t)e^{-7t}$$