Practise Problems II: MATH 316

Problem 1a)

$$\underline{\underline{a_0}} = \frac{2}{\pi}\int_0^\pi \sin(x)\,dx = \frac{2}{\pi}(-\cos x)\Big|_0^\pi = \underline{\underline{\frac{4}{\pi}}}$$

To compute a_n, we note that

$$\int_0^\pi \underbrace{\sin(x)}_{u'(x)}\underbrace{\cos(nx)}_{v(x)}\,dx = \big(-\cos(x)\big)\big(\cos(nx)\big)\Big|_0^\pi - \int_0^\pi \big(-\cos(x)\big)\big(-n\sin(nx)\big)\,dx$$

$$= \underbrace{-\cos(\pi)}_{=+1}\underbrace{\cos(n\pi)}_{(-1)^n} + 1 - \int_0^\pi \cos(x)\,n\sin(nx)\,dx$$

$$= (-1)^n + 1 - \int_0^\pi \underbrace{\cos(x)}_{u'(x)}\ \underbrace{n\sin(nx)}_{v(x)}\,dx$$

$$= (-1)^n + 1 - \cancel{\sin(x)\,n\sin(nx)\Big|_0^\pi}^{=0} + n^2\int_0^\pi \sin(x)\cos(nx)\,dx$$

We obtain for $n \geq 2$

$$(1-n^2)\int_0^\pi \sin(x)\cos(nx)\,dx = (-1)^n + 1 = \begin{cases} 2 & n = 2,4,6 \\ 0 & n = 3,5,7 \end{cases}$$

or

$$\int_0^\pi \sin(x)\cos(nx)\,dx = \begin{cases} \dfrac{-2}{n^2-1} & n = 2,4,6, \\ 0 & n = 3,5,7,\ldots \end{cases}$$

Additionally, we also have

$$\int_0^\pi \sin(x)\cos(x)\,dx = \frac{1}{2}\sin(x)^2\Big|_0^\pi = 0$$

We conclude that

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx = \begin{cases} -\frac{4}{\pi}\frac{1}{n^2-1} & n=2,4,6,\dots \\ 0 & n=1,2,\dots \end{cases}$$

Therefore, the cosine series is

$$\sin(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_{2k}\cos(2kx)$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k)^2-1}\cos(2kx)$$

Problem 16:

$$\underline{\underline{b_n}} = \frac{2}{\pi}\int_0^{\pi} 2\sin(nx)\,dx = \frac{4}{\pi}\left(-\frac{\cos(nx)}{n}\right)\Bigg|_{x=0}^{\pi}$$

$$= \frac{4}{\pi n}\left(-\underbrace{\cos(n\pi)}_{(-1)^n}+1\right) = -\frac{4}{\pi n}\left(-1+(-1)^n\right) = \underline{\underline{\begin{cases}\frac{8}{\pi n} & n=1,3,5,\dots\\ 0 & n=2,4,\dots\end{cases}}}$$

→ The Fourier sine series is

$$\underline{\underline{z = \sum_{k=1}^{\infty} \frac{8}{\pi(2k-1)} \sin((2k-1)x)}}$$

$\boxed{\text{Problem 2:}}$ Separation gives

$$\frac{T'}{4T} = \frac{X''}{X} = -\mu^2, \quad \mu \geq 0$$

Hence, $T' = -\mu^2 4T \Rightarrow T = C e^{-4\mu^2 t}$

and $\begin{cases} X'' + \mu^2 X = 0 \\ X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$

The eigenvalues and eigenfunctions are

$\underline{\mu = 0}$: $X(x) = \text{const}$ is non-trivial eigenfunction

$\underline{\mu > 0}$: $X(x) = A\sin(\mu x) + B\cos(\mu x)$

$X'(x) = \mu A\cos(\mu x) - \mu B\sin(\mu x)$

$X'(0) = 0 \Rightarrow \underline{A = 0}$

$X'(\pi) = 0 \Rightarrow \sin(\mu\pi) = 0 \Rightarrow \mu_n^2 = n^2, \quad n = 1, 2, \ldots$

$X_n = \cos(nx)$

In total: $\underline{\text{eigenvalues:}}$ $\mu_0^2 = 0$

$\mu_n^2 = n^2 \quad n = 1, 2, \ldots$

$\underline{\text{eigenfunctions:}}$ $X_0 = \frac{A_0}{2}$

$X_n = A_n\cos(nx) \quad n = 1, 2, \ldots$

The general solution is

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-4 n^2 t} \cos(nx)$$

The initial condition gives

$$u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) \stackrel{!}{=} \sin(x)$$

Hence, the coefficients are the cosine Fourier coefficients of $\sin(x)$ computed in problem 1a. We obtain:

$$\underline{u(x,t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k)^2 - 1} e^{-4 (2k)^2 t} \cos(2kx)}$$

$\boxed{\text{Problem 3:}}$ Separation gives: $u(x,t) = X(x)\,T(t)$

$$\frac{T'}{T} = 25\frac{X''}{X} + 1$$

$$\Rightarrow \frac{1}{25}\left(\frac{T'}{T} - 1\right) = \frac{X''}{X} = -\mu^2, \quad \mu \geq 0$$

in time: $\dfrac{T'}{T} = -25\mu^2 + 1 \Rightarrow T(t) = e^{(-25\mu^2+1)t}$

in space: $\left.\begin{aligned} X'' + \mu^2 X &= 0 \\ X(0) &= 0 \\ X(\pi) &= 0 \end{aligned}\right\}$

The eigenfunctions and eigenvalues are

$$\mu_n^2 = \left(\frac{n\pi}{L}\right)^2 = n^2, \quad X_n = \sin(nx), \quad n = 1, 2, \ldots$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n e^{(-25n^2+1)t} \sin(nx)$$

The coefficients b_n are the Fourier sine coefficients of $f(x) = 2$ in problem 1b. Thus, the solution is

$$\underline{u(x,t) = \sum_{k=1}^{\infty} \frac{8}{\pi(2k-1)} e^{(-25(2k-1)^2+1)t} \sin((2k-1)x)}$$

Problem 4: The steady-state solution is $\underline{v(x) = T_2}$

Setting $u(x,t) = v(x) + w(x,t)$, the function $w(x,t)$ satisfies

$$(*)\begin{cases} w_t = a^2 w_{xx} & 0 < x < L, \quad t > 0 \\ w_x(0,t) = w(L,t) = 0 \\ w(x,0) = f(x) - T_2 \end{cases}$$

Separation for $(*)$ yields

$$\frac{T'}{a^2 T} = \frac{X''}{X} = -\mu^2, \quad \mu \geq 0$$

in time: $T' = -\mu^2 a^2 T \Rightarrow T(t) = C e^{-\mu^2 a^2 t}$

in space: $\left.\begin{array}{r} X'' + \mu^2 X = 0 \\ X'(0) = 0 \\ X(L) = 0 \end{array}\right\}$

$\underline{\mu = 0}$: only the trivial solution $X \equiv 0$.

$\underline{\mu > 0}$: $X(x) = A \sin(\mu x) + B\cos(\mu x)$
$X'(x) = \mu A \cos(\mu x) - \mu B \sin(\mu x)$

$X'(0) = \mu A \overset{!}{=} 0 \Rightarrow \underline{A = 0}$

$X(L) = B\cos(\mu L) \overset{!}{=} 0 \Rightarrow \cos(\mu L) = 0$

$\Rightarrow \mu_n = \dfrac{(2n-1)\pi}{2L} \quad n = 1, 2, \ldots$

$\Rightarrow$ $\underline{\text{eigenvalues}}$: $\mu_n^2 = \left(\dfrac{(2n-1)\pi}{2L}\right)^2 \quad n = 1, 2, \ldots$

$\underline{\text{eigenfunctions}}$: $X_n = \cos\left(\dfrac{(2n-1)\pi}{2L} x\right)$

Thus, $w(x,t) = \sum_{n=1}^{\infty} a_n e^{-a^2 \mu_n^2 t} \cos(\mu_n x)$

Initial condition:

$$w(x,0) = \sum_{n=1}^{\infty} c_n \cos(\mu_n x) = f(x) - T_2$$

$$\Rightarrow \underline{\underline{c_n = \frac{2}{L} \int_0^L \left(f(x) - T_2\right) \cos\left(\frac{(2n-1)\pi}{2L} x\right) dx}}$$

The solution $u(x,t)$ now is:

$$\underline{\underline{u(x,t) = T_2 + \sum_{n=1}^{\infty} c_n e^{-a^2 \mu_n^2 t} \cos(\mu_n x)}}$$

Problem 5: We consider the lifting of the boundary conditions:

$$v(x,t) = t - xt$$

It satisfies $v(0,t) = t$ & $v(1,t) = 0$

Moreover: $v_t = 1 - x$ & $v(x,0) = 0$

$v_{xx} = 0$

Setting $u(x,t) = v(x,t) + w(x,t)$, $w(x,t)$ satisfies:

$$\left.\begin{aligned} u_t &= v_t + w_t \\ u_{xx} &= w_{xx} \end{aligned}\right\} \Rightarrow (*) \begin{cases} w_t = w_{xx} - v_t = w_{xx} - 1 + x \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = 0 \end{cases}$$

The steady-state solution of (*) satisfies:

$$\bar{v}''(x) = 1 - x, \quad 0 < x < 1$$
$$\bar{v}(0) = 0 \ \& \ \bar{v}(1) = 0$$

$\Rightarrow$

$$\bar{v}'(x) = x - \frac{x^2}{2} + A$$

$$\bar{v}(x) = \frac{x^2}{2} - \frac{x^3}{6} + Ax + B$$

The boundary conditions imply

$$\bar{v}(0) = \underline{B = 0} \quad \bar{v}(1) = \frac{1}{2} - \frac{1}{6} + A = 0 \Rightarrow \underline{\underline{A = -\frac{2}{6}}}$$

$$\Rightarrow \bar{v}(x) = \frac{x^2}{2} - \frac{x^3}{6} - \frac{2}{6}x$$

Setting $w(x,t) = \bar{v}(x) + \bar{w}(x,t)$, we obtain

$$(**)\begin{cases} \bar{w}_t = \bar{w}_{xx} \\ \bar{w}(0,t) = \bar{w}(1,t) = 0 \\ \bar{w}(x,0) = -\bar{v}(x) = -\frac{x^2}{2} + \frac{x^3}{6} + \frac{2}{6}x \end{cases}$$

The solution of $(**)$ is given by:

$$\bar{w}(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

where b_n is the Fourier sine coefficient of $\bar{v}(x)$

$$b_n = 2\int_0^1 \left(-\frac{x^2}{2} + \frac{x^3}{6} + \frac{2}{6}x\right)\sin(n\pi x)\,dx$$

In total:

$$\underline{\underline{u(x,t) = t - xt + \frac{x^2}{2} - \frac{x^3}{6} - \frac{2}{6}x + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin(n\pi x)}}$$

Problem 6

Steady-state solution $v(x)$ satisfies

$$\begin{cases} v''(x) - v = 0, & 0 < x < \pi \\ v(0) = 0 \\ v(\pi) = 1 \end{cases}$$

$$\Rightarrow v(x) = A\sinh(x) + B\cosh(x)$$

$$v(0) = B \overset{!}{=} 0 \Rightarrow \underline{\underline{B = 0}}$$

$$v(\pi) = A\sinh(\pi) = 1 \Rightarrow \underline{\underline{A = \frac{1}{\sinh(\pi)}}}$$

$$\Rightarrow \underline{\underline{v(x) = \frac{\sinh(x)}{\sinh(\pi)}}}$$

Setting $u(x,t) = v(x) + w(x,t)$, we must now solve

$$\left.\begin{aligned} u_t &\overset{!}{=} v_t + w_t = w_t \\ u_{xx} - u &= \underbrace{v'' - v}_{=0} + w_{xx} - w \end{aligned}\right\} \Rightarrow (*) \begin{cases} w_t = w_{xx} - w \\ w(0,t) = w(\pi,t) = 0 \\ w(x,0) = f(x) - v(x) \end{cases}$$

Separation yields

$$\frac{T'}{T} + 1 = \frac{X''}{X} = -\mu^2, \quad \mu \geq 0$$

in time: $T' = -(\mu^2+1)T \Rightarrow T = Ce^{-(\mu^2+1)t}$

In space: $\left.\begin{aligned} X'' + \mu^2 X &= 0 \\ X(0) &= 0 \\ X(\pi) &= 0 \end{aligned}\right\}$ eigenvalues: $\mu_n^2 = n^2$, $n = 1, 2, \ldots$

eigenfunctions: $X_n = \sin(nx)$

Hence, the solution is

$$\underline{\underline{u(x,t) = \frac{\sinh(x)}{\sinh(\pi)} + \sum_{n=1}^{\infty} c_n e^{-(n^2+1)t} \sin(nx)}}$$

where c_n is the Fourier sine coefficient of $f(x) - v(x)$:

$$c_n = \frac{2}{\pi} \int_0^{\pi} \left(f(x) - v(x)\right) \sin(nx)\, dx$$