Problem 1:

$\underline{\mu = 0}$: $y''(x) = 0 \Rightarrow y(x) = Ax + B$

$y'(x) = A$

B.C. $y'(0) = 0 \Rightarrow \underline{A = 0}$

$y(1) = 0 \Rightarrow \underline{B = 0}$

$\underline{\mu = 0 \text{ is not an eigenvalue}}$

$\underline{\mu > 0}$: $y(x) = A\cos(\mu x) + B\sin(\mu x)$

$y'(x) = -A\mu\sin(\mu x) + B\mu\cos(\mu x)$

$y'(0) = 0 \Rightarrow \underline{B = 0}$

$y(1) = 0 \Rightarrow A\cos(\mu) = 0$

Hence, $\mu_n = \frac{2n-1}{2}\pi$, $n = 1, 2, \ldots$

$\underline{\text{Eigenvalues: } \mu_n^2 = \left(\frac{2n-1}{2}\pi\right)^2, n = 1, 2,}$

$\underline{\text{Eigenfunctions: } X_n = \cos\left(\frac{2n-1}{2}\pi x\right)}$

Problem 2:

The steady-state problem is

$$\begin{cases} 0 = v''(x) + \sin(x), & 0 < x < \pi \\ 0 = v(0) \\ 0 = v(\pi) \end{cases}$$

The solution is $\quad v''(x) = -\sin(x)$

$$\Rightarrow v'(x) = \cos(x) + A$$

$$\Rightarrow v(x) = \sin(x) + Ax + B$$

Boundary conditions: $\underline{\underline{v(x) = \sin(x)}}$

Writing $u(x,t) = w(x,t) + v(x)$, $w(x,t)$ satisfies

$$\begin{cases} w_t = w_{xx}, & 0 < x < \pi, \quad t > 0 \\ w(0,t) = 0, \quad w(\pi,t) = 0 \\ w(x,0) = -v(x) = -\sin(x) \end{cases}$$

The solution is

$$\underline{\underline{w(x,t) = -\sin(x)\, e^{-t}}}$$

Hence, $\underline{\underline{u(x,t) = \sin(x)\left(1 - e^{-t}\right)}}$

Problem 3:

Separation gives

$$T' X = 4 T X'' - T X$$

$$\Rightarrow \frac{T'}{T} = 4 \frac{X''}{X} - 1 \Rightarrow \frac{1}{4}\left(\frac{T'}{T} + 1\right) = \frac{X''}{X} = -\mu^2 \qquad \mu \geq 0$$

eigenvalue problem:

$$\begin{cases} X'' + \mu^2 X = 0 \\ X'(0) = X'(1) = 0 \end{cases}$$

eigenvalues / eigenfunctions are $\begin{cases} \mu_0 = 0, \ X_0 = \frac{A_0}{2} \\ \mu_n^2 = n^2\pi^2; \\ X_n = A_n \cos(n\pi x) \end{cases} \quad n = 1, 2, \dots$

in time: $T' + T = -4\mu^2 T \Leftrightarrow T = -(4\mu^2 + 1) T$

$$\Rightarrow T(t) = C e^{-(4\mu^2+1)t}$$

$$\left(= C e^{-t} \text{ for } \mu = 0 \right)$$

Hence, the general form of the solution is:

$$\underline{\underline{u(x,t) = \frac{A_0}{2} e^{-t} + \sum_{n=1}^{\infty} A_n e^{-(4n^2\pi^2+1)t} \cos(n\pi x)}}$$

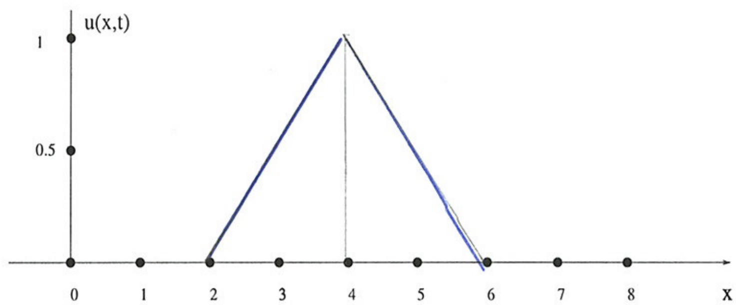
The coefficients $A_0, A_1, \dots, A_n, \dots$ are determined by the Fourier cosine coefficients of $f(x)$.

Problem 4 (10 points)

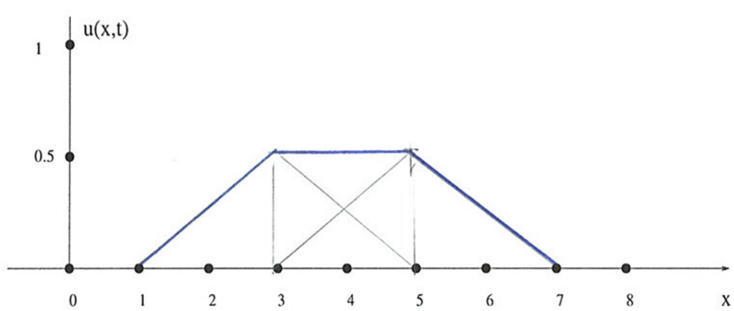
Consider the wave equation:

$$u_{tt} = u_{xx}, \qquad 0 < x < 8,\ t > 0,$$
$$u(0,t) = 0, \quad u(8,t) = 0,$$
$$u(x,0) = f(x) = \begin{cases} -1 + \frac{x}{2} & 2 \le x \le 4, \\ 3 - \frac{x}{2} & 4 \le x \le 6, \\ 0 & \text{otherwise.} \end{cases}$$
$$u_t(x,0) = 0.$$

In the coordinate systems provided below, carefully sketch the solution $u(x,t)$ at $t = 0$, $t = 1$, $t = 2$, and $t = 3$.



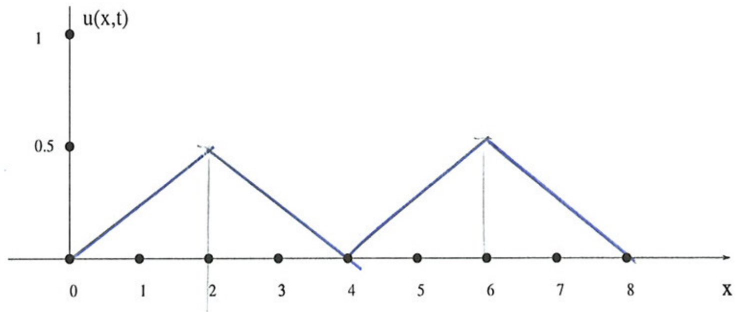
a) $t = 0$



b) $t = 1$

Problem 4 (continued)

c) $t = 2$



d) $t = 3$