SOLUTION: MIDTERM 2, Mar 18, 2009

$\boxed{\text{Problem 1:}}$

a) $\underline{b_n} = 2 \int_0^1 \frac{x}{2} \sin(n\pi x)\, dx = \int_0^1 \underbrace{x}_{u(x)} \underbrace{\sin(n\pi x)}_{v'(x)}\, dx$

$$= x\left(-\frac{\cos(n\pi x)}{n\pi}\right)\Bigg|_0^1 - \int_0^1 \left(-\frac{\cos(n\pi x)}{n\pi}\right) dx$$

$$= -\frac{\cos(n\pi)}{n\pi} + \int_0^1 \frac{\cos(n\pi x)}{n\pi}\, dx$$

$\cos(n\pi) = (-1)^n$ for $n = 1, 2, \dots$

$$= -\frac{\cos(n\pi)}{n\pi} + \underbrace{\cancel{\frac{\sin(n\pi x)}{(n\pi)^2}\Bigg|_0^1}}_{=0} = \underline{\underline{\frac{(-1)^{n+1}}{n\pi}}}$$

$$\Rightarrow \underline{\underline{\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(n\pi x)}}$$

b) The steady-state solution $v(x)$ satisfies:

$$0 = v''(x) - 4v(x) \implies v(x) = A\cosh(2x) + B\sinh(2x)$$

The boundary conditions $v(0) = 0$ and $v(1) = 1$ imply:

$v(0) = A \overset{!}{=} 0 \Rightarrow \underline{A = 0}$

$v(1) = B\sinh(2) \overset{!}{=} 1 \Rightarrow \underline{\underline{B = \frac{1}{\sinh(2)}}}$

$$\Rightarrow \underline{\underline{v(x) = \frac{\sin(2x)}{\sinh(2)}}}$$

c) Look for solution $u(x,t)$ of the form $u(x,t) = X(x)T(t)$

Separation yields
$$XT' = X''T - 4XT$$

$$\iff \frac{T'}{T} + 4 = \frac{X''}{X} = -\mu^2, \quad \mu \geq 0$$

In time :

$$T' = -(\mu^2 + 4)\,T \quad \Rightarrow \quad \underline{\underline{T(t) = C e^{-(\mu^2+4)t}}}$$

In space :

$$\left.\begin{array}{r} X'' + \mu^2 X = 0 \\ X(0) = 0 \\ X(1) = 0 \end{array}\right\} \quad \begin{array}{l} \text{eigenvalues: } \mu^2 = n^2\pi^2 \qquad n = 1, 2, 3, \ldots \\ \text{eigenfunctions: } X_n = \sin(n\pi x) \end{array}$$

The solution is

$$\underline{\underline{u(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi}\, e^{-(n^2\pi^2+4)t} \sin(n\pi x)}}$$

where we have used the Fourier sine series in 1a)

Problem 2

a) look for a particular solution of the form

$$v(x,t) = Ax^2 + Bx + Ct$$

Then: $v_{xx} = 2A$, $v_t = C$ $\Big\} \Rightarrow \underline{C = 2A}$

$$v(x,t) = Ax^2 + Bx + 2At \Rightarrow v_x(x,t) = 2Ax + B$$

The boundary conditions then imply

$$v_x(0,t) = \underline{B \overset{!}{=} 1}$$

$$v_x(\pi,t) \overset{!}{=} 0 = 2A\pi + 1 \Rightarrow \underline{A = -\frac{1}{2\pi}}$$

$$\Rightarrow \underline{v(x,t) = -\frac{1}{2\pi}x^2 + x - \frac{1}{\pi}t}$$

b) $u(x,t) = X(x)T(t)$

$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} = -\mu^2, \quad \mu \geq 0$$

In time: $T' = -\mu^2 T \Rightarrow T = Ce^{-\mu^2 t}$

In space: $X'' + \mu^2 X = 0, \quad X'(0) = X'(\pi) = 0$

eigenvalues: $\mu_n^2 = n^2$, $n = 0, 1, 2, \ldots$ eigenfunctions: $X_n = \cos(nx)$, $n = 0, 1, 2, \ldots$

Since $\cos(2x)$ is already expressed in terms of cosine eigenfunctions the solution is $\underline{u(x,t) = e^{-4t}\cos(2x)}$