Name:______________________ Student Number:______________________

**Problem 1:** (total 20pt) Consider the heat conduction problem

$$u_t = 4u_{xx}, \quad 0 < x < \pi, \ t > 0$$
$$u(0,t) = \alpha, \quad u(\pi, t) = \beta$$
$$u(x,0) = 2\sin(3x)$$

(a) Find the steady state solution $v(x)$ for $\alpha = 2$ and $\beta = 4$.

(b) Find the solution $u(x,t)$ for $\alpha = 0$ and $\beta = 0$.

(a) $0 = v_{xx} \Rightarrow v(x) = a x + b$

BC: $v(0) = b \overset{!}{=} 2$

$v(\pi) = a\pi + 2 \overset{!}{=} 4 \quad \Rightarrow a = \frac{2}{\pi}$

$\Rightarrow \quad v(x) = \frac{2}{\pi} x + 2$

(b) $u_t = 4u_{xx}$

$u(0,t) = 0 \quad , \quad u(\pi, t) = 0$

$u(x,0) = 2\sin(3x)$

Assume $u(x,t) = X(x)\, T(t)$

$\dot{T} X = 4 X'' T$

$\Rightarrow \quad \frac{\dot{T}}{4T} = \frac{X''}{X} =: -\lambda^2$

Time: $\dot{T} = -\lambda^2 4\, T \quad \Rightarrow \boxed{T(t) = c\, e^{-4\lambda^2 t}}$ see next page

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extra page problem 1

Space: $X'' + \lambda^2 X = 0$ , $X(0) = 0$ , $X(\pi) = 0$

$\lambda = 0$: trivial solution

$\lambda > 0$: $X(x) = A \cos \lambda x + B \sin \lambda x$

$X(0) = A \overset{!}{=} 0$ , $X(\pi) = B \sin(\lambda\pi) \overset{!}{=} 0$

$B = 0$ : trivial solution $\Rightarrow B \neq 0$

$\sin(\lambda\pi) = 0$

$\Rightarrow \lambda_n = n$ , $n = 1, 2, 3, \ldots$

eigenvalues: $\lambda_n^2 = n^2$ , $n = 1, 2, 3, \ldots$

eigenfunctions: $X_n = \sin(nx)$ , $n = 1, 2, 3, \ldots$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2 t} \sin(nx)$$

$$\Rightarrow u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) \overset{!}{=} 2 \sin 3x$$ (Initial condition)

$$\Rightarrow b_n = \begin{cases} 2 & \text{if } n = 3 \\ 0 & \text{if } n \neq 3 \end{cases}$$

$$\Rightarrow u(x,t) = 2 \sin(3x)\, e^{-36t}$$

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**Problem 2**: (total 20pt) Find the solution of the heat conduction problem

$$u_t = 16u_{xx} - u, \quad 0 < x < 1, \ t > 0$$
$$u_x(0,t) = 0, \quad u_x(1,t) = 0$$
$$u(x,0) = x$$

Assume $u(x,t) = X(x)T(t)$

$$X\dot{T} = 16X''T - XT = (16X'' - X)T$$

$$\Leftrightarrow \quad \frac{\dot{T}}{T} = 16\frac{X''}{X} - 1 \quad \Leftrightarrow \frac{1}{16}\left(\frac{\dot{T}}{T} + 1\right) = \frac{X''}{X} =: -\lambda^2$$

Time: $\dot{T} = -(16\lambda^2 + 1)T$

$$\Rightarrow \quad T(t) = c\, e^{-(16\lambda^2+1)t}$$

Space: $X'' + \lambda^2 X = 0$

$\lambda = 0$: $X(x) = ax + b$

$X'(x) = a \quad \Rightarrow \quad \left.\begin{array}{l} X'(0) = a \overset{!}{=} 0 \\ X'(1) = a \overset{!}{=} 0 \end{array}\right\} \Rightarrow X(x) = b \ (= \text{const})$

eigenvalue: $\lambda = 0$, eigenfunction $X_0 = 1$

$\lambda > 0$ $\quad X(x) = A\cos\lambda x + B\sin\lambda x$

$$X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$$

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extra page problem 2

BC: $X'(0) = B\lambda \overset{!}{=} 0 \Rightarrow B = 0$ since $\lambda > 0$

$X'(1) = -A\lambda \sin(\lambda) \overset{!}{=} 0$

$A \neq 0$ otherwise we get the trivial solution (and $\lambda > 0$)

$\Rightarrow \sin(\lambda) = 0$

$\Rightarrow \lambda_n = n\pi \quad , n = 1, 2, \ldots$

eigenvalues: $\lambda_n^2 = (n^2\pi^2) \quad , n = 1, 2, \ldots$

eigenfunctions: $X_n = \cos(n\pi x) \quad , n = 1, 2, \ldots$

The general solution is:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(16n^2\pi^2)t} \cos(n\pi x)$$

Initial condition:

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \overset{!}{=} x$$

Hence the coefficients are the cosine Fourier coefficients of $x$.

$$a_0 = 2\int_0^1 x \, dx = 1$$

$$a_n = 2\int_0^1 x\cos(n\pi x)\,dx = \underbrace{2\left[x \sin(n\pi x)\frac{1}{n\pi}\right]_0^1}_{\frac{2}{n\pi}\sin(n\pi) = 0 \text{ if } n \text{ integer}} - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\,dx$$

$$a_n = -\frac{2}{n\pi}\int_0^1 \sin(n\pi x)\, dx$$

$$= \left[\frac{2}{(n\pi)^2}\cos(n\pi x)\right]_0^1 = \frac{2}{(n\pi)^2}\left[\cos(n\pi) - 1\right]$$

$$= \frac{2}{(n\pi)^2}\left((-1)^n - 1\right)$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}\left((-1)^n - 1\right)\cos(n\pi x)\, e^{-(16n^2\pi^2+1)t}$$