MATH 257/316 MIDTERM 2 SECTION 104 NOV 2008

1. $u_{tt} = c^2 u_{xx} + e^{-t}\sin 5x \qquad 0<x<\pi/2 \qquad (1)$

BC: $u(0,t)=0 \qquad u_x(\frac{\pi}{2},t)=t$

IC: $u(x,0)=0 \qquad u_t(x,0) = x+\sin 3x$

LET US CONSTRUCT A FUNCTION $w(x,t)$ THAT SATISFIES THE INHOMOGENEOUS BC. ASSUME $w(x,t)=\alpha(t)+\beta(t)x \qquad w_x=\beta(t)$

NOW $0=w(0,t)=\alpha(t) \qquad \therefore t = w_x(\pi/2,t)=\beta(t) \Rightarrow w(x,t)=xt$

NOW LET $u(x,t)=w(x,t)+v(x,t)$ THEN SUBSTITUTING INTO (1) WE OBTAIN

$$u_{tt} = \cancel{w_{tt}}+v_{tt} = c^2(\cancel{w_{xx}}+v_{xx})+e^{-t}\sin 5x \Rightarrow v_{tt}=c^2v_{xx}+e^{-t}\sin 5x$$

BC: $0=u(0,t)=w(0,t)+v(0,t)=0+v(0,t) \Rightarrow v(0,t)=0$

$\cancel{t}=u_x(\pi/2,t)=w_x(\pi/2,t)+v_x(\pi/2,t)=\cancel{t}+v_x(\pi/2,t) \Rightarrow v_x(\pi/2,t)=0.$

IC: $0=u(x,0)=w(x,0)+v(x,0)=0+v(x,0) \Rightarrow v(x,0)=0$

$\cancel{x}+\sin 3x = u_t(x,0)=w_t(x,0)+v_t(x,0)=\cancel{x}+v_t(x,0) \Rightarrow v_t(x,0)=\sin 3x. \qquad (2)$

SINCE PROBLEM (2) FOR $v(x,t)$ HAS HOMOGENEOUS BC WE CAN DEFINE EIGENFUNCTIONS AND EIGENVALUES BY SEPARATING VARIABLES (EXCLUDING THE FORCING TERM $e^{-t}\sin 5x$)

THE APPROPRIATE EIGENVALUE PROBLEM IS: $X''+\lambda^2X=0 \quad X(0)=0=X'(\pi/2)$

$\therefore X(x)=A\cos\lambda x+B\sin\lambda x \qquad X'=-A\lambda\sin\lambda x+B\lambda\cos\lambda x$

$X(0)=A=0 \qquad X'(\pi/2)=B\lambda\cos(\lambda\pi/2)=0 \Rightarrow \lambda_n\pi/2=(2n+1)\frac{\pi}{2} \Rightarrow \boxed{\lambda_n=(2n+1),\ n=0,1,\ldots \quad X_n=\sin\lambda_n x.}$

NOW ASSUME AN EIGENFUNCTION EXPANSION FOR $v(x,t)$

$$v(x,t)=\sum_{n=0}^{\infty}\hat v_n(t)\sin\lambda_n x \qquad v_{tt}=\sum_{n=0}^{\infty}\frac{d^2\hat v_n}{dt^2}\sin\lambda_n x \qquad v_{xx}=\sum_{n=0}^{\infty}(-\lambda_n^2)\hat v_n\sin\lambda_n x.$$

$$\therefore v_{tt}-c^2v_{xx}-e^{-t}\sin 5x = \sum_{n=0}^{\infty}\left\{\frac{d^2\hat v_n}{dt^2}+c^2\lambda_n^2\hat v_n-e^{-t}\delta_{n2}\right\}\sin\lambda_n x=0.$$

$$\frac{d^2\hat v_n}{dt^2}+c^2\lambda_n^2\hat v_n=e^{-t}\delta_{n2}$$

~~TO OBTAIN A PARTICULAR SOLUTION ASSUME~~ $\hat v_n = Ae^{-t} \Rightarrow +Ae^{-t}+c^2\lambda_n^2Ae^{-t}=e^{-t}\delta_{n2}.$

$\therefore A=\left(\frac{\delta_{n2}}{1+\lambda_n^2c^2}\right)$ SO THE GEN. SOL'N IS: $\hat v_n = \left(\frac{\delta_{n2}}{1+\lambda_n^2c^2}\right)+A_n\cos\lambda_nct+B_n\sin\lambda_nct)$

$$\therefore v(x,t)=\sum_{n=0}^{\infty}\left\{\left(\frac{\delta_{n2}e^{-t}}{1+\lambda_n^2c^2}\right)+A_n\cos\lambda_nct+B_n\sin\lambda_nct\right\}\sin\lambda_n x$$

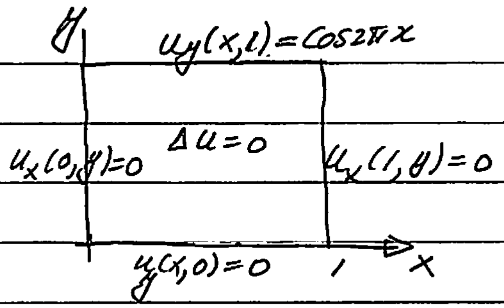
$$0=v(x,0)=\sum_{n=0}^{\infty}\left[\left(\frac{\delta_{n2}}{1+\lambda_n^2c^2}\right)+A_n\right]\sin\lambda_n x \Rightarrow A_2=\frac{-1}{1+\lambda_2^2c^2} \qquad A_n=0 \quad n\neq 2$$

$$\sin(3x)=v_t(x,0)=\sum_{n=0}^{\infty}\left[\left(-\frac{\delta_{n2}}{1+\lambda_n^2c^2}\right)+B_n\lambda_nc\right\}\sin\lambda_n x \Rightarrow B_1=\frac{1}{\lambda_1c},\ B_2=\frac{1}{c\lambda_2(1+\lambda_2^2c^2)}$$

$$B_n=0 \quad n\neq 1,2$$

$$\therefore u(x,t)=xt+\frac{1}{1+\lambda_2^2c^2}\left[e^{-t}-\cos(\lambda_2ct)+\frac{\sin(\lambda_2ct)}{\lambda_2c}\right]\sin\lambda_2x+\frac{1}{\lambda_1c}\sin\lambda_1ct\sin\lambda_1x.$$

2. $\Delta u = u_{xx} + u_{yy} \qquad 0<x,y<1$



LET $u(x,y) = X(x)Y(y)$

$$\frac{X''}{X(x)} = \frac{-Y''(y)}{Y(y)} = -\lambda^2$$

$$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X'(1) \end{array}\right\} \quad \lambda_n = n\pi \quad n = 0,1,\ldots \quad X_n(x) \in \{1, \cos\lambda_n x\}$$

$\underline{\lambda_0 = 0}$: $\quad Y_0'' = 0 \quad Y_0 = A_0 + B_0 y \quad Y_0'(0) = B_0 = 0 \quad Y_0 = A_0.$

$\underline{\lambda_n > 0}$: $\left.\begin{array}{l} Y_n'' - \lambda_n^2 Y_n = 0 \\ Y_n'(0) = 0 \end{array}\right\} \quad \begin{array}{l} Y_n = A_n \cosh\lambda_n y + B_n \sinh\lambda_n y \\ Y_n' = A_n\lambda_n \sinh\lambda_n y + B_n \lambda_n \cosh\lambda_n y \end{array}$

$$Y_n'(0) = B_n\lambda_n = 0 \quad B_n = 0.$$

$$\therefore u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh\lambda_n y \cos\lambda_n x$$

$$u_y(x,y) = 0 + \sum_{n=1}^{\infty} A_n \lambda_n \sinh\lambda_n y \cos\lambda_n x$$

$$f(x) = \cos(2\pi x) = u_y(x,1) = 0 + \sum_{n=1}^{\infty} \{A_n\lambda_n \sinh\lambda_n\} \cos(n\pi x)$$

FIRSTLY WE NOTE THAT

$$\int_0^1 \cos(2\pi x)\,dx = 0$$

SO A STEADY SOLUTION EXISTS.

MATCHING COEFFICIENTS $\quad A_2 = \dfrac{1}{(2\pi)\sinh(2\pi)}, \quad A_n = 0 \quad n \neq 2 \quad n \geq 1.$

$$\therefore u(x,y) = A_0 + \frac{1}{(2\pi)} \frac{\cosh(2\pi y)}{\sinh(2\pi)} \cos(2\pi x)$$

IF $\displaystyle\int_0^1\int_0^1 u(x,y,t=0)\,dx\,dy = 100 = \int_0^1\int_0^1 \left\{A_0 + \frac{1}{(2\pi)} \frac{\cosh(2\pi y)\cos(2\pi x)}{\sinh(2\pi)}\right\} dx\,dy$

$$\therefore A_0 = 100$$

IN WHICH CASE

$$u(x,y) = 100 + \frac{1}{(2\pi)} \frac{\cosh(2\pi y)}{\sinh(2\pi)} \cos(2\pi x).$$