MATH 257/316 MIDTERM 2 SECTION 104 NOVEMBER 2007

1. $$u_t = u_{xx} - x \qquad (1)$$

BC: $u(0,t) = t \quad u(1,t) = 0 \qquad (1b)$

IC: $u(x,0) = \sin\pi x + \frac{x}{2}(x-1) \qquad (1c)$

FIRSTLY LET US FIND A FUNCTION $W(x,t)$ THAT SATISFIES THE INHOMOG BC (1b): $W(x,t) = t(1-x)$ DOES THE TRICK.

NOW LET $u(x,t) = W(x,t) + V(x,t)$

$$u_t = W_t + V_t = (1-\cancel{x}) + V_t = \cancel{W_{xx}} + V_{xx} - \cancel{x} \Rightarrow V_t = V_{xx} - 1 \qquad (2)$$

$$\cancel{t} = u(0,t) = W(0,t) + V(0,t) = \cancel{t} + V(0,t) \Rightarrow V(0,t) = 0$$

$$0 = u(1,t) = W(1,t) + V(1,t) = 0 + V(1,t) \Rightarrow V(1,t) = 0 \qquad (2b)$$

$$\sin\pi x + \frac{x}{2}(x-1) = u(x,0) = W(x,0) + V(x,0) = 0 + V(x,0) \Rightarrow V(x,0) = \sin\pi x + \frac{x}{2}(x-1)$$

NOW LET V^∞ BE THE TO THE STEADY STATE VERSION OF (2) & (2b)

$$0 = V^\infty_{xx} - 1 \Rightarrow V^\infty_{xx} = 1, \quad V^\infty_x = x + A, \quad V^\infty = \frac{x^2}{2} + Ax + B$$

$$0 = V^\infty(0) = B, \quad 0 = V^\infty(1) = \frac{1}{2} + A \Rightarrow A = -\frac{1}{2} \qquad V^\infty(x) = -\frac{x}{2}(1-x)$$

NOW LET $V(x,t) = V^\infty(x) + \omega(x,t)$

$$V_t = (\cancel{V^\infty(x)} + \omega)_t = \cancel{V^\infty_{xx}} + \omega_{xx} - \cancel{1} \Rightarrow \omega_t = \omega_{xx}$$

$$0 = V(0,t) = V^\infty(0) + \omega(0,t) = 0 + \omega(0,t) \Rightarrow \omega(0,t) = 0$$

$$0 = V(1,t) = V^\infty(1) + \omega(1,t) = 0 + \omega(1,t) \Rightarrow \omega(1,t) = 0.$$

$$\sin\pi x + \frac{x}{2}(\cancel{x-1}) = V(x,0) = \cancel{\frac{x}{2}(x-1)} + \omega(x,0) \Rightarrow \omega(x,0) = \sin(\pi x)$$

SEPARATING VARIABLES $\omega(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 t}\sin(\lambda_n x)$

WHERE $\lambda_n = (n\pi) \quad n = 1, 2, \ldots$

$$\therefore \quad \sin\pi x = \omega(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x), \Rightarrow A_1 = 1, \; A_n = 0 \;\; n \geq 2$$

$$\therefore \quad \omega(x,t) = e^{-\pi^2 t}\sin(\pi x)$$

PUTTING ALL THE SOLUTIONS TOGETHER

$$u(x,t) = t(1-x) + \frac{x}{2}(x-1) + e^{-\pi^2 t}\sin(\pi x)$$

2. $u_{tt} = c^2 u_{xx} + x \qquad 0<x<1$

BC: $u(0,t) = 0 = u(1,t)$

IC: $u(x,0) = \frac{1}{6c^2} x(1-x^2) \qquad \frac{\partial u(x,0)}{\partial t} = \sin \pi x$

LOOK FOR A STEADY SOLUTION: $W(x)$: $0 = c^2 W_{xx} + x$

$$W_{xx} = -\frac{x}{c^2} \qquad W_x = -\frac{x^2}{2c^2} + A \qquad W = -\frac{x^3}{6c^2} + Ax + B$$

$$W(0) = B = 0 \qquad W(1) = -\frac{1}{6c^2} + A = 0 \Rightarrow A = \frac{1}{6c^2} \Rightarrow W(x) = \frac{x}{6c^2}(1-x^2)$$

NOW LET $u(x,t) = W(x) + V(x,t)$

$$u_{tt} = (\cancel{W} + V)_{tt} = c^2(\cancel{W_{xx}} + V_{xx}) + \cancel{x} \qquad \Rightarrow \qquad V_{tt} = c^2 V_{xx}.$$

$$0 = u(0,t) = W(0) + V(0,t) = 0 + V(0,t) \qquad \Rightarrow \qquad V(0,t) = 0$$

$$0 = u(1,t) = W(1) + V(1,t) = 0 + V(1,t) \qquad \Rightarrow \qquad V(1,t) = 0$$

$$\cancel{\frac{1}{6c^2} x(1-x^2)} = u(x,0) = W(x) + V(x,0) = \cancel{\frac{1}{6c^2} x(1-x^2)} + V(x,0) \Rightarrow V(x,0) = 0$$

$$\sin \pi x = u_t(x,0) = \frac{\partial}{\partial t} \cancel{W(x)} + V_t(x,0) \qquad \Rightarrow \qquad V_t(x,0) = \sin \pi x.$$

METHOD 1: USING D'ALEMBERT'S SOLUTION $V(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(\pi s)\, ds$

$$\therefore u(x,t) = \frac{1}{6c^2} x(1-x^2) + \frac{1}{2c}\left[-\frac{\cos \pi s}{\pi}\right]_{x-ct}^{x+ct}$$

$$= \frac{1}{6c^2} x(1-x^2) + \frac{1}{2\pi c}\left[\cos \pi(x-ct) - \cos \pi(x+ct)\right]$$

METHOD 2: SEPARATING VARIABLES $V(x,t) = \sum_{n=1}^{\infty} [A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)] \sin(\lambda_n x)$

$$0 = V(x,0) = \sum_{n=1}^{\infty} A_n \sin \lambda_n x \quad \Rightarrow A_n = 0$$

$$\sin \pi x = V_t(x,0) = \sum_{n=1}^{\infty} B_n(\lambda_n c) \sin(n\pi x) \Rightarrow B_n = 0 \quad n \neq 1$$

$$B_1 = \frac{1}{\lambda_1 c} = \frac{1}{\pi c}.$$

$$\therefore \quad V(x,t) = \frac{1}{\pi c} \sin(\pi c t) \sin(\pi x) = \frac{1}{2\pi c}\left[\cos \pi(x-ct) - \cos \pi(x+ct)\right]$$

AS SHOWN ABOVE

$$u(x,t) = \frac{1}{6c^2} x(1-x^2) + \frac{1}{\pi c} \sin(\pi c t) \sin \pi x.$$