MATH 257/316 MIDTERM 2 SECTION 101 SOLUTIONS NOV 2008

1. $u_t = u_{xx} + e^{-t}\left(\sin\left(\frac{3\pi x}{2}\right) - x\right) \quad 0<x<1 \quad t>0$

$u(0,t)=0 \quad u_x(1,t)=e^{-t}$

$u(x,0) = \sin\left(\frac{\pi x}{2}\right) + x$

TO REMOVE THE INHOMOGENEOUS BC LET $W(x,t) = \alpha(t) + \beta(t)x$

$W(0,t) = \alpha(t) = 0 \quad W_x(x,t) = \beta(t) = e^{-t} \Rightarrow W(x,t) = xe^{-t}.$

NOW LET $u(x,t) = W(x,t) + V(x,t)$

$u_t = W_t + V_t = -\cancel{x}e^{-t} + V_t = \cancel{W_{xx}} + V_{xx} + e^{-t}\left(\sin\left(\frac{3\pi x}{2}\right) - \cancel{x}\right)$

$\therefore \quad V_t = V_{xx} + e^{-t}\sin\left(\frac{3\pi x}{2}\right).$

BC: $0 = u(0,t) = \cancel{W(0,t)} + V(0,t) \Rightarrow V(0,t)=0$

$\cancel{e^{-t}} = u_x(1,t) = W_x(1,t) + V_x(1,t) = \cancel{e^{-t}} + V_x(1,t) \Rightarrow V_x(1,t)=0.$

IC: $\cancel{x} + \sin\left(\frac{\pi x}{2}\right) = u(x,0) = W(x,0) + V(x,0) = \cancel{x} + V(x,0) \Rightarrow V(x,0) = \sin\left(\frac{\pi x}{2}\right).$

THE EIGENVALUE PROBLEM FOR THE HOMOGENEOUS BC IS

$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X'(1) \end{array}\right\} \quad \lambda_n = \frac{(2n+1)\pi}{2} \quad n=0,1,\ldots \quad X_n = \sin(\lambda_n x)$

NOW LET $V(x,t) = \sum_{n=0}^{\infty} \hat{V}_n(t)\sin(\lambda_n x), \quad V_t = \sum_{n=0}^{\infty} \frac{d\hat{V}}{dt}\sin\lambda_n x, \quad V_{xx} = \sum_{n=0}^{\infty}\left(-\lambda_n^2 \hat{V}_n \sin\lambda_n x\right)$

$\therefore V_t - V_{xx} - e^{-t}\sin\left(\frac{3\pi x}{2}\right) = \sum_{n=0}^{\infty}\left\{\frac{d\hat{V}_n}{dt} + \lambda_n^2\hat{V}_n - e^{-t}\delta_{n1}\right\}\sin\lambda_n x = 0.$

$\therefore \quad \frac{d\hat{V}_n}{dt} + \lambda_n^2\hat{V}_n = e^{-t}\delta_{n1} \Rightarrow \frac{d}{dt}\left(e^{\lambda_n^2 t}\hat{V}_n\right) = e^{(\lambda_n^2-1)t}\delta_{n1}$

$\therefore \quad \hat{V}_n = \frac{e^{-t}\delta_{n1}}{(\lambda_n^2-1)} + b_n e^{-\lambda_n^2 t}$

$\therefore \quad V(x,t) = \sum_{n=0}^{\infty}\left\{\frac{e^{-t}\delta_{n1}}{\lambda_n^2-1} + b_n e^{-\lambda_n^2 t}\right\}\sin\lambda_n x$

$\sin\left(\frac{\pi x}{2}\right) = V(x,0) = \sum_{n=0}^{\infty}\left\{\frac{\delta_{n1}}{\lambda_n^2-1} + b_n\right\}\sin\lambda_n x$

MATCHING COEFFICIENTS:

$b_0 = 1 \quad b_1 = \frac{-1}{\left(\frac{3\pi}{2}\right)^2 - 1} \quad b_n = 0 \quad n \geq 2.$

$\therefore u(x,t) = xe^{-t} + e^{-\left(\frac{\pi}{2}\right)^2 t}\sin\left(\frac{\pi x}{2}\right) + \frac{1}{\left(\frac{3\pi}{2}\right)^2 - 1}\left\{e^{-t} - e^{-\left(\frac{3\pi}{2}\right)^2 t}\right\}\sin\left(\frac{3\pi x}{2}\right).$

2. LET $W(x) = x$ WHICH SATISFIES THE x BOUNDARY CONDITIONS.

THEN LET $u(x,y) = W(x) + V(x,y)$

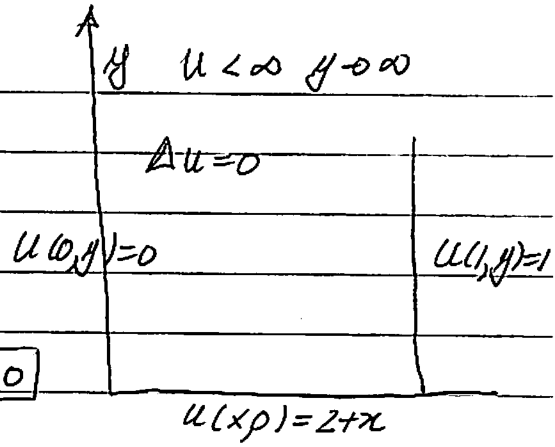
$$0 = \Delta u = \cancel{\Delta W}^{0} + \Delta V \Rightarrow \boxed{\Delta V = 0}$$

$$0 = u(0,y) = \cancel{W(0)}^{0} + V(0,y) \Rightarrow \boxed{V(0,y) = 0}$$

$$\cancel{1}^{0} = u(1,y) = W(1) + V(1,y) = \cancel{1}^{0} + V(1,y) \Rightarrow \boxed{V(1,y) = 0}$$

$$u(x,0) = 2 + x = W(x) + V(x,0) = x + V(x,0)$$

$$\therefore \boxed{V(x,0) = 2}.$$



NOW SEPARATE VARIABLES $V(x,y) = X(x)\,Y(y) \Rightarrow \dfrac{X''}{X(x)} = -\dfrac{Y''}{Y(y)} = -\lambda^2$

$$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(1) \end{array}\right\} \Rightarrow \begin{array}{l} \lambda_n = n\pi \quad n = 1, 2, \ldots \\ X_n = \sin(n\pi x) \end{array}$$

$$Y'' - \lambda^2 Y = 0 \Rightarrow Y_n = A_n e^{\lambda_n y} + B_n e^{-\lambda_n y}$$

SINCE $|u| < \infty$ AS $y \to \infty$ WE REQUIRE $A_n = 0$.

$$\therefore u(x,y) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n y} \sin \lambda_n x$$

$$u(x,0) = 2 = \sum_{n=1}^{\infty} B_n \sin \lambda_n x \qquad B_n = \frac{2}{1}\int_0^1 2 \sin \lambda_n x \, dx$$

$$= 4\left[-\frac{\cos(n\pi x)}{(n\pi)}\right]_0^1$$

$$= 4\left[\frac{1 - (-1)^n}{n\pi}\right]$$

$$= \frac{8}{\pi}\,\frac{1}{(2k+1)} \qquad k = 0, 1, \ldots$$

$$\therefore u(x,y) = \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{e^{-(2k+1)\pi y}}{(2k+1)} \sin(2k+1)\pi x$$