MATH 257/316 MIDTERM 2 SECTION 101 NOVEMBER 2007.

1. $$u_t = u_{xx} \qquad 0 < x < 1$$

$$\text{BC:}\quad u(0,t) = 0 \qquad u(1,t) = t$$

$$\text{IC:}\quad u(x,0) = \frac{1}{6}x(x^2-1)$$

LET $W(x,t)$ BE A FUNCTION THAT SATISFIES THE INHOMOGENEOUS BC:

$$W(x,t) = xt$$

NOW LET $u(x,t) = W(x,t) + V(x,t)$

$$u_t = W_t + V_t = x + V_t = u_{xx} = \cancelto{0}{W_{xx}} + V_{xx} \Rightarrow \quad V_t = V_{xx} - x$$

$$0 = u(0,t) = W(0,t) + V(0,t) = 0 + V(0,t) \Rightarrow \quad V(0,t) = 0$$

$$\cancel{t} = u(1,t) = W(1,t) + V(1,t) = \cancel{t} + V(1,t) \Rightarrow \quad V(1,t) = 0$$

$$\frac{1}{6}x(x^2-1) = u(x,0) = W(x,0) + V(x,0) = 0 + V(x,0) \Rightarrow \quad V(x,0) = \frac{1}{6}x(x^2-1)$$

NOW LOOK FOR A STEADY STATE SOLUTION TO THE V EQ & BC:

$$0 = V^\infty_{xx} - x \Rightarrow V^\infty_{xx} = x \Rightarrow V^\infty_x = \frac{x^2}{2} + A \Rightarrow V^\infty = \frac{x^3}{6} + Ax + B$$

$$0 = V^\infty(0) = B \qquad 0 = V^\infty(1) = \frac{1}{6} + A \Rightarrow A = -\frac{1}{6}$$

$$\therefore V^\infty(x) = \frac{x}{6}(x^2-1)$$

NOW LET $V(x,t) = V^\infty(x) + \omega(x,t)$

$$V_t = \cancelto{0}{V^\infty_t} + \omega_t = V^\infty_{xx} + \omega_{xx} - x \Rightarrow \quad \omega_t = \omega_{xx}$$

$$0 = V(0,t) = V^\infty(0) + \omega(0,t) \Rightarrow \quad \omega(0,t) = 0$$

$$0 = V(1,t) = V^\infty(1) + \omega(1,t) \Rightarrow \quad \omega(1,t) = 0$$

$$\frac{1}{6}x(x^2-1) = V(x,0) = V^\infty(x) + \omega(x,0) = \frac{1}{6}x(x^2-1) + \omega(x,0) \Rightarrow \quad \omega(x,0) = 0.$$

SINCE THE EQ, THE IC & BC FOR ω ARE ALL HOMOGENEOUS $\omega \equiv 0$

$$\therefore \quad u(x,t) = xt + \frac{x}{6}(x^2-1)$$

2. $u_{tt} = c^2 u_{xx} + \sin \pi x$ (1)

BC: $u(0,t) = 0 \qquad u(1,t) = 0$ (1b)

IC: $u(x,0) = 0 \qquad u_t(x,0) = \sin(3\pi x)$

LOOK FOR A STEADY STATE SOLUTION OF (1) & (1b): $0 = c^2 W_{xx} + \sin(\pi x)$

$$W_{xx} = -\frac{1}{c^2}\sin \pi x \qquad W_x = +\frac{1}{c^2\pi}\cos \pi x + A \qquad W = \frac{+\sin \pi x}{\pi^2 c^2} + Ax + B$$

$$W(0) = B = 0 \qquad W(1) = \cancel{\frac{\sin \pi}{\pi^2 c^2}} + A = 0 \Rightarrow A = 0$$

$$\therefore \quad W(x) = \frac{\sin \pi x}{\pi^2 c^2}$$

NOW LET $u(x,t) = W(x) + V(x,t)$

$$u_{tt} = \cancel{W_{tt}} + V_{tt} = c^2(\cancel{W_{xx}} + V_{xx}) + \sin \pi x \Rightarrow \quad V_{tt} = c^2 V_{xx}$$

$$0 = u(0,t) = W(0) + V(0,t) = 0 + V(0,t) \Rightarrow \quad V(0,t) = 0$$

$$0 = u(1,t) = W(1) + V(1,t) = 0 + V(1,t) \Rightarrow \quad V(1,t) = 0$$

$$0 = u(x,0) = \frac{\sin(\pi x)}{\pi^2 c^2} + V(x,0) \Rightarrow \quad V(x,0) = -\frac{\sin(\pi x)}{\pi^2 c^2}$$

$$\sin 3\pi x = \frac{\partial u}{\partial t}(x,0) = \cancel{\frac{\partial}{\partial t}(W(x))}^{\,0} + V_t(x,0) \Rightarrow \quad V_t(x,0) = \sin(3\pi x)$$

METHOD 1: BY D'ALEMBERTS SOL'N:

$$V(x,t) = \frac{-1}{2\pi^2 c^2}\left[\sin \pi(x+ct) + \sin \pi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(3\pi s)\,ds$$

$$= -\frac{1}{2\pi^2 c^2}\left[\sin \pi(x+ct) + \sin \pi(x-ct)\right] + \frac{1}{6\pi c}\left[\cos 3\pi(x-ct) - \cos 3\pi(x+ct)\right]$$

METHOD 2: SEPARATING VARIABLES $V(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)\right]\sin \lambda_n x$

WHERE $\lambda_n = n\pi \quad n = 1,\ldots;\quad -\frac{\sin(\pi x)}{\pi^2 c^2} = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \Rightarrow A_1 = \frac{-1}{\pi^2 c^2} \quad A_n = 0 \quad n \geq 2$

$$V_t(x,0) = \sin(3\pi x) = \sum_{n=1}^{\infty} B_n(\lambda_n c)\sin(n\pi x) \Rightarrow B_3 = \frac{1}{3\pi c} \quad B_n = 0 \quad n \neq 3.$$

$$\therefore \quad V(x,t) = \frac{-1}{\pi^2 c^2}\cos \pi ct \sin(\pi x) + \frac{1}{3\pi c}\sin(3\pi ct)\sin(3\pi x)$$

$$= -\frac{1}{2\pi^2 c^2}\left[\sin \pi(x+ct) + \sin \pi(x-ct)\right] + \frac{1}{6\pi c}\left[\cos 3\pi(x-ct) - \cos 3\pi(x+ct)\right]$$

$$\therefore \quad u(x,t) = \frac{\sin \pi x}{\pi^2 c^2} - \frac{1}{2\pi^2 c^2}\left[\sin \pi(x+ct) + \sin \pi(x-ct)\right] + \frac{1}{6\pi c}\left[\cos 3\pi(x-ct) - \cos 3\pi(x+ct)\right]$$