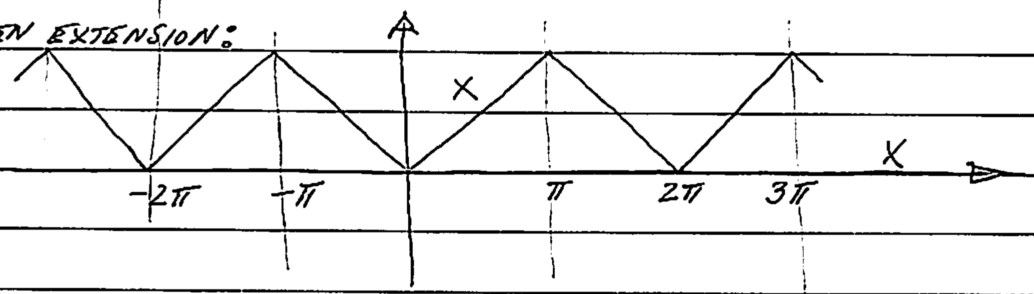
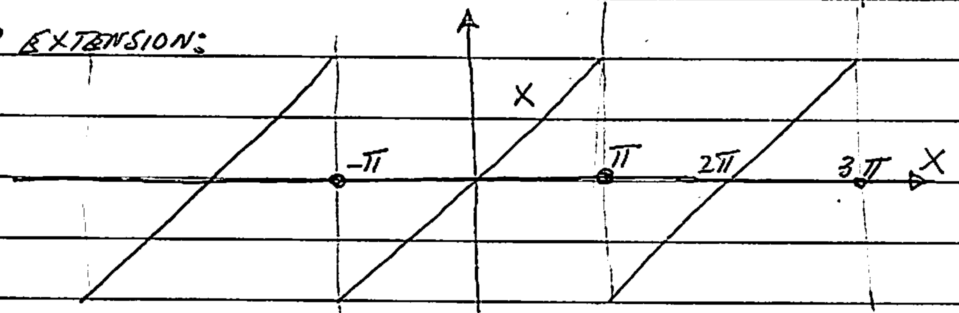
MATH 257/316 MIDTERM 1 SECTION 104 OCT 2008

1. (a) $f(x) = x$

EVEN EXTENSION:



ODD EXTENSION:



(b) $f_{EVEN}(x) = |x|$ ON $[-\pi, \pi]$ $\qquad |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\begin{cases} 0 & n \text{ EVEN} \\ \frac{-2}{n^2} & n \text{ ODD} \end{cases} = \frac{-2}{(2k+1)^2} \quad k = 0, 1, \ldots$$

$$\therefore \quad |x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2} \qquad (*)$$

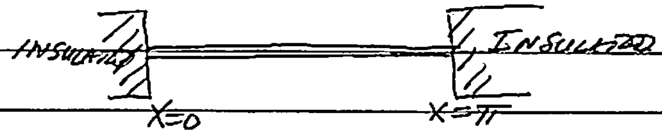
OBSERVE THAT $g(x) = \frac{\pi}{2} - |x| = \frac{\pi}{2} - \left\{ \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2} \right\}$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2}$$

(c) IF WE LET $x = 0$ IN $(*)$ THEN

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

2. $u_t = u_{xx}$



$u_x(0,t) = 0 = u_x(\pi,t)$

$u(x,0) = 2\sin^2 x = 1 - \cos 2x$

LET $u(x,t) = X(x)\,T(t)$

$\dot{T}(t)\,X(x) = X''(x)\,T(t)$

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \quad \text{CONST}$$

$\dot{T} = -\lambda^2 T \Rightarrow \quad T = C e^{-\lambda^2 t}$

$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X'(\pi) \end{array}\right\}$ $\quad X = A\cos\lambda x + B\sin\lambda x \qquad X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$X'(0) = B\lambda = 0 \qquad B = 0$ OR $\lambda = 0$.

$X'(\pi) = -A\lambda\sin(\lambda\pi) = 0 \Rightarrow \lambda_n = n \quad n = 0, 1, \ldots$

$\therefore$ EIGENVALUES ARE $\lambda_n = n$ $n = 0, 1, \ldots$ & CORRESPONDING EIGENFUNCTIONS $X_n = \cos nx$

$$\therefore \quad u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t}\cos(nx)$$

IMPOSE INITIAL CONDITION: $u(x,0) = 1 - \cos 2x$

$$\therefore \quad 1 - \cos 2x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

BY INSPECTION WE CAN MATCH COEFFICENTS SINCE $\{1, \cos nx\}$ ARE INDEPENDENT WE CONCLUDE $a_0 = 2$ AND $a_2 = -1$ AND THE REST ARE 0.

ALTERNATIVELY

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(1 - \cos 2x)\,dx = \frac{2}{\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(1 - \cos 2x)\cos(nx)\,dx = \frac{2}{\pi}\left\{\int_0^{\pi} 1.\cos nx\,dx - \int_0^{\pi}\cos 2x\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left[\underset{\text{BY ORTOGONALITY}}{0} - \left\{\begin{array}{ll} 0 & n \neq 2 \\ \pi/2 & n = 2 \end{array}\right\}\right]$$

$$= -1$$

$$\therefore \quad u(x,t) = 1 - e^{-4t}\cos 2x$$

3. $Ly = 2x^2 y'' + 3xy' - (1+x^2)y = 0$ $\quad P(x) = 2x^2 \quad Q(x) = 3x \quad R(x) = -(1+x^2)$

(a) SINCE $P(x) = 2x^2 > 0$ FOR ALL $x > 0$, ALL $x > 0$ ARE ORDINARY PTS.

SINCE $P(0) = 0$ $x = 0$ AND $R(0) \neq 0$ $\quad x = 0$ IS A SINGULAR PT.

$\lim\limits_{x \to 0} x \frac{Q(x)}{P(x)} = \lim\limits_{x \to 0} x \frac{(3x)}{2x^2} = \frac{3}{2} < \infty$; $\lim\limits_{x \to 0} \frac{x^2(-1-x^2)}{2x^2} = -\frac{1}{2} < \infty \Rightarrow x = 0$ IS A RSP.

THE INDICIAL EQ IS: $r(r-1) + \frac{3}{2} r - \frac{1}{2} = 0 \therefore (2r-1)(r+1) = 0 \quad r = 1/2, -1$

(b) SINCE $x = 0$ IS A RSP WE ASSUME A FROBENIUS EXPANSION OF THE FORM:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \overset{n=m}{=} \sum_{m=2}^{\infty} a_m x^{m+r} + a_0 x^r + a_1 x^{r+1}$$

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{n+r+2} \overset{n=m-2}{=} \sum_{m=2}^{\infty} a_{m-2} x^{m+r}$$

$$x y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} \overset{n=m}{=} \sum_{m=2}^{\infty} a_m (m+r) x^{m+r} + a_0 r x^r + a_1 (r+1) x^{r+1}$$

$$x^2 y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} \overset{n=m}{=} \sum_{m=2}^{\infty} a_m (m+r)(m+r-1) x^{m+r} + a_0 r(r-1) x^r + a_1 (r+1) r x^{r+1}$$

LET $m = n+2 \quad n = m-2 \quad n = 0 \Rightarrow m = 2$

$$0 = Ly = \sum_{m=2}^{\infty} \left[a_m (m+r)[2(m+r-1)+3] - a_m - a_{m-2} \right] x^{m+r} + a_0 \{2r(r-1)+3r-1\} x^r + a_1 \{2r(r+1)+3(r+1)-1\} x^{r+1}$$

$x^r >$ $\quad a_0 (2r^2 + r - 1) = a_0 (2r-1)(r+1) = 0 \quad r = 1/2$ OR $r = -1$.

$x^{r+1} >$ $\quad a_1 \{2r^2 + 5r + 2\} = 0$ SINCE $\{\ \} \neq 0$ FOR $r = 1/2, -1 \Rightarrow a_1 = 0$.

$x^{m+r} >$ $\quad a_m [(m+r)(2(m+r)+1) - 1] - a_{m-2} = 0$

$\therefore \quad a_m = a_{m-2} / \{(m+r)[2(m+r)+1] - 1\} \qquad a_1 = 0 \Rightarrow a_3 = a_5 = \cdots = 0$

$r = -1$: $\quad a_m = a_{m-2} / \{(m-1)(2m-1) - 1\} = a_{m-2} / (2m^2 - 3m)$

$a_2 = a_0/2$; $\quad a_4 = a_2/20 = a_0/40, \ldots$

$$y_1(x) = a_0 x^{-1} \{1 + x^2/2 + x^4/40 + \cdots\}$$

$r = 1/2$: $\quad a_m = a_{m-2} / \{(m+1/2)(2m+1+1) - 1\} = a_{m-2} / [(2m+1)(m+1) - 1] = a_{m-2} / \{2m^2 + 3m\}$

$a_2 = a_0/14$; $\quad a_4 = a_2/44 = a_0/(14 \cdot 44)$; ..

$$y_2(x) = a_0 x^{1/2} \{1 + x^2/14 + x^4/(44 \cdot 14) + \cdots\}$$