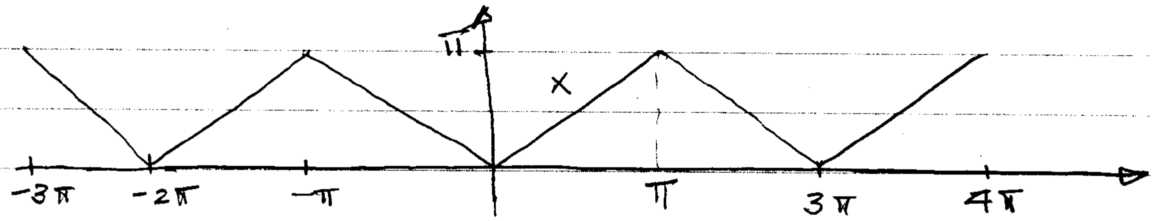
MATH 257/316 MIDTERM1 SECTION 104 SOLUTIONS 17 OCT '07

1. (a)



$L = \pi$

(b) $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\,\frac{x^2}{2}\Big|_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{2}{\pi}\begin{cases} 0 & n \text{ IS EVEN} \\ -2/n^2 & n \text{ ODD.} \end{cases}$$

$$\therefore\quad f(x) = x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos((2m+1)x)}{(2m+1)^2}$$

2. $u_t = u_{xx} \quad 0<x<\pi,\ t>0$

BC: $u_x(0,t) = 0 = u(\pi,t)$

IC: $u(x,0) = 3\cos\left(\frac{5x}{2}\right)$

LET $u(x,t) = X(x)T(t) \Rightarrow \frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 = \text{CONST}$

$\dot{T} = -\lambda^2 T \Rightarrow T(t) = Ce^{-\lambda^2 t}$

$X'' + \lambda^2 X = 0 \Rightarrow X = A\cos\lambda x + B\sin\lambda x \quad X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$X'(0) = B\lambda = 0 \Rightarrow B = 0$ OR $\lambda = 0$; $X(\pi) = A\cos\lambda\pi = 0 \Rightarrow \boxed{\lambda_n = \frac{(2n+1)}{2}},\ n = 0, 1, \ldots$

IF $\lambda = 0$ WE OBTAIN THE TRIVIAL SOLUTION $X \equiv 0$ SO WE EXCLUDE $\lambda = 0$

THE CORRESPONDING EIGENFUNCTIONS ARE $X_n = \cos\left(\frac{2n+1}{2}x\right)$

$$\therefore\quad u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{(2n+1)x}{2}\right)e^{-\left(\frac{2n+1}{2}\right)^2 t}$$

APPLYING IC:

$$3\cos\left(\frac{5x}{2}\right) = u(x,0) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{(2n+1)x}{2}\right) = A_0\cos\left(\frac{x}{2}\right) + A_1\cos\left(\frac{3x}{2}\right) + A_2\cos\left(\frac{5x}{2}\right) + \cdots$$

MATCHING COEFFICIENTS $A_2 = 3$ & $A_n = 0 \quad n \neq 2$ OR BY DIRECT CALCULATION

$$\int_0^{\pi}\cos\left(\frac{5x}{2}\right)\cos\left(\frac{(2n+1)x}{2}\right)dx = \begin{cases} 0 & n \neq 2 \\ \pi/2 & n = 2 \end{cases}$$ WHICH YIELD THE SAME RESULT

$$\therefore\quad u(x,t) = 3e^{-\left(\frac{5}{2}\right)t}\cos\left(\frac{5x}{2}\right)$$ IS THE SOLUTION

3. $Ly = 4x^2y'' - (x^2+x)y' + y = 0$

a) $x > 0$ ARE ALL ORDINARY POINTS

$x = 0$ IS A REGULAR SINGULAR POINT SINCE

$$\lim_{x\to 0} x\left(\frac{-(x^2+x)}{4x^2}\right) = -\frac{1}{4} = p_0 < \infty, \quad \lim_{x\to 0} x^2\left(\frac{1}{4x^2}\right) = \frac{1}{4} = q_0 < \infty.$$

b) LET $y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{m=1}^{\infty} a_m x^{m+r} + a_0 x^r$

$$xy' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} = \sum_{m=1}^{\infty} a_m (m+r) x^{m+r} + a_0 r x^r$$

$$x^2y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r+1} = \sum_{m=1}^{\infty} a_{m-1} (m+r-1) x^{m+r}$$

$$x^2y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} = \sum_{m=1}^{\infty} a_m (m+r)(m+r-1) x^{m+r} + a_0 r(r-1) x^r$$

LET $m+\cancel{r} = n+\cancel{r}+1 \Rightarrow n = m-1 \quad n=0 \Rightarrow m=1$

$$\therefore Ly = \sum_{m=1}^{\infty} \left[a_m \left[4(m+r)(m+r-1) - (m+r) + 1\right] - a_{m-1}(m+r-1)\right] x^{m+r}$$

$$+ \, a_0 \left[4r(r-1) - r + 1\right] x^r$$

$x^r]$ $\quad a_0(4r^2 - 5r + 1) = a_0(4r-1)(r-1) = 0 \quad r = 1, \; r = 1/4.$

$x^{m+r}]$ $$a_m = \frac{+(m+r-1)\, a_{m-1}}{(m+r)(4(m+r-1)-1)+1}$$

<u>$r=1$:</u> $$a_m = \frac{m\, a_{m-1}}{(m+1)(4m-1)+1}$$

$$a_1 = \frac{1.a_0}{2.3+1} = \frac{a_0}{7}; \quad a_2 = \frac{2a_1}{3.7+1} = \frac{a_1}{11} = \frac{a_0}{77}$$

$$y_1(x) = a_0 x \left[1 + \frac{x}{7} + \frac{x^2}{77} + \cdots\right]$$

<u>$r = 1/4$:</u> $$a_m = \frac{(m-3/4)\, a_{m-1}}{(m+1/4)(4m-3-1)+1} = \frac{(m-3/4)\, a_{m-1}}{(4m+1)(m-1)+1}$$

$$a_1 = \frac{(1/4)\, a_0}{5.0+1} = \frac{a_0}{4} \qquad a_2 = \frac{(5/4)\, a_1}{9.1+1} = \frac{1}{8} a_1 = \frac{a_0}{32}$$

$$y_2(x) = a_0 x^{1/4}\left[1 + \frac{x}{4} + \frac{x^2}{32} + \cdots\right]$$