1 a) P, Q, R polynomials

10

$\Rightarrow \quad x \in \mathbb{R}$ singular $\Rightarrow P(x) = 9x^2 = 0$

$\Rightarrow x = 0$

, x is not a common factor of P, Q, R

$\Rightarrow \quad x_0 = 0$ is singular 1

$$\lim_{x \to 0} \frac{x Q(x)}{P(x)} = \lim_{x \to 0} \frac{x \, 9x}{9x^2} = 1 \text{ , finite}$$

$$\lim_{x \to 0} \frac{x^2 R(x)}{P(x)} = \lim_{x \to 0} \frac{-x^2(1+x)}{9x^2} = -\frac{1}{9} \text{ , finite}$$

2

$\Rightarrow$ $\boxed{x_0 = 0 \text{ is a regular singular point}}$

$\boxed{\forall x \in (0, \infty), \ x \text{ is an ordinary point}}$ 2

point at ∞,

let $z = \frac{1}{x}$

then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\left(\frac{-1}{x^2}\right) = -z^2 \frac{dy}{dz}$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right) = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$

$$= \frac{d^2y}{dz^2}\left(\frac{-1}{x^2}\right)^2 + \frac{dy}{dz}\left(\frac{2}{x^3}\right)$$

$$= z^4 \frac{d^2y}{dz^2} + 2z^3 \frac{dy}{dz}$$

thus $Ly = 9x^2 \frac{d^2y}{dx^2} + 9x \frac{dy}{dx} - (1+x)y = 0$

$$= \frac{9}{z^2}\left(z^4 \frac{d^2y}{dz^2} + 2z^3 \frac{dy}{dz}\right)$$

$$+ \frac{9}{z}(-z^2)\frac{dy}{dz} - \left(1 + \frac{1}{z}\right)y = 0$$

$$= 9z^2 \frac{d^2y}{dz^2} + 9z \frac{dy}{dz} - \left(1 + \frac{1}{z}\right)y = 0$$

$$\lim_{z \to 0} \frac{z^2 R(x)}{P(x)} = \lim_{z \to 0} \frac{-z^2\left(1+\frac{1}{z}\right)}{9z^2} = \lim_{z \to 0}\left(\frac{-1}{9} - \frac{1}{9z}\right) \text{ DNE}$$

5

$\Rightarrow$ $\boxed{x = \infty \text{ is an irregular sing. point.}}$

b) $\frac{}{5}$

$y(1) = 5$

$y'(1) = 0$

here $x_0 = 1$ which is ordinary ①

$\Rightarrow$ assume Taylor series centred at 1,

i.e. $\boxed{y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n}$ ②

furthermore, $a_0 = 5$, $a_1 = 0$ (don't need this)

The function, $\frac{9x}{9x^2} = \frac{1}{x}$ is singular at 0, thus the radius of convergence of the Taylor series of this function centred at 1 is $|0 - 1| = 1$.

The function $\frac{-(1+x)}{9x^2}$ is also singular at 0, thus corresponding radius is also 1,

Then the radius of convergence

of $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n \geq \min\{1, 1\}$

$= \boxed{1}$ ②

c) ~~35~~

Corresponding Euler eqn: $9x^2y'' + 9xy' - y = 0$

indicial eqn: $9r(r-1) + 9r - 1 = 0$

$$9r^2 = 1$$

$$r = \pm\tfrac{1}{3} \quad 3$$

which are real, distinct & do not differ by an integer

$\Rightarrow \exists$ two linearly indep. solns of the form

$$y_1(x) = x^{\frac{1}{3}}\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)$$

$$y_2(x) = x^{-\frac{1}{3}}\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)$$

METHOD 1

(30) $r = \frac{1}{3}$,

$$y = x^{\frac{1}{3}} + a_1 x^{\frac{4}{3}} + a_2 x^{\frac{7}{3}} + \dots \quad 2$$

$$y' = \tfrac{1}{3}x^{-\frac{2}{3}} + \tfrac{4}{3}a_1 x^{\frac{1}{3}} + \tfrac{7}{3}a_2 x^{\frac{4}{3}} + \dots \quad 2$$

$$y'' = -\tfrac{2}{9}x^{-\frac{5}{3}} + \tfrac{4}{9}a_1 x^{-\frac{2}{3}} + \tfrac{28}{9}a_2 x^{\frac{1}{3}} + \dots \quad 2$$

plug into ODE:

$$\begin{aligned}
9x^2y'' + 9xy' - y - xy
&= \cancel{-2x^{\frac{1}{3}}} + 4a_1 x^{\frac{4}{3}} + 28a_2 x^{\frac{7}{3}} + \dots \\
&\quad \cancel{+3x^{\frac{1}{3}}} + 12a_1 x^{\frac{4}{3}} + 21a_2 x^{\frac{7}{3}} + \dots \\
&\quad \cancel{-x^{\frac{1}{3}}} - a_1 x^{\frac{4}{3}} - a_2 x^{\frac{7}{3}} + \dots \quad 5 \\
&\quad - x^{\frac{4}{3}} - a_1 x^{\frac{7}{3}} + \dots \\
&= 0
\end{aligned}$$

$$\Rightarrow 15a_1 - 1 = 0 \Rightarrow \boxed{a_1 = \tfrac{1}{15}} \quad 2$$

$$\& \quad 48a_2 - a_1 = 0 \Rightarrow \boxed{a_2 = \tfrac{1}{15 \cdot 48}} \quad 2$$

$\underline{r = -\frac{1}{3}}$

$$y = x^{-\frac{1}{3}} + a_1 x^{\frac{2}{3}} + a_2 x^{\frac{5}{3}} + \dots \quad (2)$$

$$y' = -\tfrac{1}{3} x^{-\frac{4}{3}} + \tfrac{2}{3} a_1 x^{-\frac{1}{3}} + \tfrac{5}{3} a_2 x^{\frac{2}{3}} + \dots \quad (2)$$

$$y'' = \tfrac{4}{9} x^{-\frac{7}{3}} - \tfrac{2}{9} a_1 x^{-\frac{4}{3}} + \tfrac{10}{9} a_2 x^{-\frac{1}{3}} + \dots \quad (2)$$

plug into ODE:

$$9x^2y'' + 9xy' - y - xy$$
$$= \cancel{4x^{-\frac{1}{3}}} - 2a_1 x^{\frac{2}{3}} + 10 a_2 x^{\frac{5}{3}} + \dots$$
$$\cancel{-3x^{-\frac{1}{3}}} + 6a_1 x^{\frac{2}{3}} + 15 a_2 x^{\frac{5}{3}} + \dots$$
$$\cancel{-x^{-\frac{1}{3}}} - a_1 x^{\frac{2}{3}} - a_2 x^{\frac{5}{3}} + \dots \quad (5)$$
$$- x^{\frac{2}{3}} - a_1 x^{\frac{5}{3}} + \dots$$
$$= 0$$

$$\Rightarrow \quad 3a_1 - 1 = 0 \quad \Rightarrow \quad \boxed{a_1 = \tfrac{1}{3}} \quad (2)$$
$$\& \quad 24 a_2 - a_1 = 0 \quad \Rightarrow \quad \boxed{a_2 = \frac{1}{3 \cdot 24}} \quad (2)$$

$\boxed{\text{or}}$

METHOD 2

(30) $$p_0 + p_1 x + p_2 x^2 + \dots = \frac{xQ(x)}{P(x)} = 1$$
$$\Rightarrow p_0 = 1 \quad \& \quad p_n = 0, \ \forall n \geq 1 \quad (4)$$

$$q_0 + q_1 x + q_2 x^2 + \dots = \frac{x^2R(x)}{P(x)} = -\tfrac{1}{9} - \tfrac{1}{9}x$$
$$\Rightarrow q_0 = -\tfrac{1}{9}, \quad q_1 = -\tfrac{1}{9}, \quad q_n = 0, \ \forall n \geq 2 \quad (4)$$

$$F(r) = r^2 - \tfrac{1}{9}$$

then $$a_n(r) = \frac{-\sum_{k=0}^{n-1} a_k(r)\big(\underbrace{(r+k)p_{n-k}}_{\text{always } 0} + \underbrace{q_{n-k}}_{= \begin{cases} -\frac{1}{9}, & k=n-1 \\ 0, & \text{otherwise} \end{cases}}\big)}{F(r+n)} \quad (4)$$

$$= \frac{-a_{n-1}(r)\left(-\frac{1}{9}\right)}{(r+n)^2 - \frac{1}{9}}$$

$$= \frac{a_{n-1}(r)}{9(r+n)^2 - 1} \quad (4)$$

$\underline{r = \frac{1}{3}}$ $\quad a_0 = 1$ (1)

$$a_1 = \frac{1}{9\left(\frac{4}{3}\right)^2 - 1} = \frac{1}{15} \quad (3)$$

$$a_2 = \frac{a_1}{9\left(\frac{7}{3}\right)^2 - 1} = \frac{1}{15 \cdot 48} \quad (3)$$

$\underline{r = -\frac{1}{3}}$ $\quad a_0 = 1$ (1)

$$a_1 = \frac{1}{9\left(\frac{2}{3}\right)^2 - 1} = \frac{1}{3} \quad (3)$$

$$a_2 = \frac{a_1}{9\left(\frac{5}{3}\right)^2 - 1} = \frac{1}{3 \cdot 24} \quad (3)$$

via either method, two linearly indep. solns are:

$$y_1(x) = x^{\frac{1}{3}}\left(1 + \frac{1}{15}x + \frac{1}{15 \cdot 48}x^2 + \dots\right)$$

$$y_2(x) = x^{-\frac{1}{3}}\left(1 + \frac{1}{3}x + \frac{1}{3 \cdot 24}x^2 + \dots\right)$$

(2)

② 30

$$u_t = u_{xx}$$

write $u(x,t) = X(x)T(t)$ ②

$$\Rightarrow X T' = X'' T$$

$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} = k \in \mathbb{R}, \text{ constant}$$ ②

$$X'' - kX = 0$$
$$X(0) = 0$$
$$X(\pi) = 0$$ ②

$k>0$ $\Rightarrow$ $X(x) = a_1 e^{\sqrt{k}x} + a_2 e^{-\sqrt{k}x}$

$X(0) = 0 \Rightarrow a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$

$X(\pi) = 0 \Rightarrow a_1(e^{\sqrt{k}\pi} - e^{-\sqrt{k}\pi}) = 0 \Rightarrow a_1 = 0$

$\Rightarrow$ trivial soln ②

$k=0$ $\Rightarrow$ $X(x) = a_1 + a_2 x$

$X(0) = 0 \Rightarrow a_1 = 0$

$X(\pi) = 0 \Rightarrow a_2\pi = 0 \Rightarrow a_2 = 0$

$\Rightarrow$ trivial soln ②

$k = -\lambda^2 < 0$ $\Rightarrow X(x) = a_1\cos(\lambda x) + a_2\sin(\lambda x)$ ②

$\Rightarrow X(0) = 0 \Rightarrow a_1 = 0$ ②

$\Rightarrow X(\pi) = 0 \Rightarrow a_2\sin(\lambda\pi) = 0$

$\Rightarrow \lambda\pi = n\pi, \ n \in \mathbb{Z}$ ②

$$\Rightarrow \boxed{\lambda_n = n, \quad n = 1, 2, 3, \ldots}$$ ②

$$\boxed{X_n(x) = \sin(nx)}$$ ②

$$T' - kT = 0$$

$$T_n' + n^2 T = 0$$

$$\Rightarrow \boxed{T_n(t) = e^{-n^2 t}} \quad 3$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} c_n \sin(nx) e^{-n^2 t} \quad 2$$

$$u(x,0) = x(\pi - x) = \sum_{n=1}^{\infty} c_n \sin(nx)$$

$$\Rightarrow c_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin(nx)\, dx$$

$$= \frac{2}{\pi} \frac{2}{n^3} \left(1 - \cos(n\pi)\right)$$

$$= \frac{4}{n^3 \pi} \left(1 - (-1)^n\right) = \begin{cases} \dfrac{8}{n^3 \pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \quad 5$$

thus $$\boxed{u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{8}{n^3 \pi} \sin(nx) e^{-n^2 t}}$$