MATH 257/316 MIDTERM 1 SECTION 103 11 OCT 2006.

$$\left.\begin{aligned} &u_t = \alpha^2 u_{xx} \\ &BC:\ u(0,t) = 0 = u(L,t) \\ &IC:\ u(x,0) = f(x) \end{aligned}\right\} \quad (1)$$

LET $u(x,t) = X(x)T(t)$

$$X(x)\dot{T}(t) = \alpha^2 X''(x)T(t)$$

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 = \text{CONST}$$

$$\dot{T}(t) = -\alpha^2\lambda^2 T(t) \Rightarrow \quad T(t) = Ce^{-\lambda^2\alpha^2 t}$$

$$\left.\begin{aligned} &X'' + \lambda^2 X = 0 \\ &X(0) = 0 = X(L) \end{aligned}\right\} \quad \begin{aligned} &X(x) = A\cos\lambda x + B\sin(\lambda x) \\ &X(0) = A = 0 \end{aligned}$$

$$X(L) = B\sin(\lambda L) = 0 \overset{B\neq 0}{\Rightarrow} \lambda L = n\pi \quad n = 1, 2, \ldots$$

$\therefore$ EIGENVALUES ARE $\lambda_n = \frac{n\pi}{L}$ $n = 1, 2, \ldots$ AND CORRESPONDING EIGENFUNCTIONS ARE $X_n = \sin\left(\frac{n\pi x}{L}\right)$

THUS BY SUPERPOSITION THE GENERAL SOLUTION FOR THE LINEAR EQ (1) IS:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)$$

IC:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

WHERE $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \delta_{mn}\frac{L}{2} \Rightarrow B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

FOR $L=1$ $f(x) = x$: $B_n = 2\int_0^1 x\sin(n\pi x)dx = 2\left\{-\frac{x\cos(n\pi x)}{(n\pi)}\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)dx\right.$

$$= 2\left[-\frac{\cos(n\pi)}{n\pi} + \frac{\sin(n\pi x)}{\cancel{(n\pi)^2}}\Big|_0^1\right] = \frac{2(-1)^{n+1}}{n\pi}$$

$$\therefore \quad f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{L}\right)$$

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n e^{-\alpha^2(n\pi)^2 t}\sin\left(\frac{n\pi x}{L}\right)$$

2. $Ly = 2x^2 y'' + x y' - (1+x) y = 0 \quad (1)$

(a) ALL POINTS $x_0 > 0$ ARE ORDINARY POINTS OF (1).

$x = 0$: REWRITE EQ IN THE FORM

$$y'' + \frac{x}{2x^2} y' - \frac{(1+x)}{2x^2} y = 0$$

NOW $\quad x\left(\frac{x}{2x^2}\right) \xrightarrow{x \to 0} \frac{1}{2} \qquad x^2\left(-\frac{(1+x)}{2x^2}\right) \xrightarrow{x \to 0} -\frac{1}{2}$

$\therefore$ $x = 0$ IS A REGULAR SINGULAR POINT.

(b) TO CONSIDER THE POINT AT ∞ LET $t = 1/x$

$$\frac{d}{dx} = \left(\frac{d}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{d}{dt} = -t^2\frac{d}{dt}$$

$$\frac{d^2}{dx^2} = -t^2\frac{d}{dt}\left(-t^2\frac{d}{dt}\right) = t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt}$$

$$Ly = \frac{2}{t^2}\left(t^4\ddot{y} + 2t^3\dot{y}\right) + \frac{1}{t}\left(-t^2\dot{y}\right) - \left(1 + \frac{1}{t}\right)y = 0$$

$$2t^2\ddot{y} + (4t - t)\dot{y} - \left(\frac{t+1}{t}\right)y = 0$$

$$\ddot{y} + \frac{3}{t}\dot{y} - \frac{(t+1)}{t^3}y$$

SINCE $\quad t^2\left[-\frac{(t+1)}{t^3}\right] \to -\infty$ AS $t \to 0$ THE POINT AT ∞

IS AN IRREGULAR SINGULAR POINT.

(c) RE-WRITE EQ IN THE FORM

$$2[(x-1)^2 + 2(x-1) + 1]\, y'' + [(x-1) + 1]\, y' - [(x-1) + 2]\, y = 0$$

THEN LET $t = x - 1 \quad \frac{d}{dx} = \frac{d}{dt}$

$$\therefore\ 2[t^2 + 2t + 1]\,\ddot{y} + [t + 1]\,\dot{y} - (t+2)\,y = 0.$$

SINCE $t = 0$ IS AN ORDINARY POINT ASSUME A SERIES SOLUTION OF THE FORM $\quad y(t) = \sum_{n=0}^{\infty} a_n t^n$

GATHER THE LIKE POWERS IN t AND EQUATE THE COEFFICIENTS OF THESE POWERS TO OBTAIN A RECURSION FOR a_n. FROM THE RECURSION WE OBTAIN 2 INDEPENDENT SOLUTIONS.

(d) $x=0$ IS A RSP FOR $Ly = 2x^2y'' + xy' - (1+x)y = 0$

SO ASSUME A SOLUTION OF THE FORM $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$0 = Ly = \sum_{n=0}^{\infty} \left[2a_n(n+r)(n+r-1) + a_n(n+r) - a_n\right] x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1}$$

$m = n+1,\ n=0 \Rightarrow m=1$

$$0 = a_0\{2r(r-1) + r - 1\}x^r + \sum_{m=1}^{\infty} \{a_m[(m+r)(2m+2r-1)-1] - a_{m-1}\} x^{m+r}$$

$x^r >$ $a_0 \neq 0 \Rightarrow (2r+1)(r-1) = 0$ INDICIAL EQ $\Rightarrow r = -1/2 \quad r = 1$

$x^{m+r} >$ $$a_m = \frac{a_{m-1}}{(m+r)(2(m+r)-1)-1}$$

$r = -1/2$: $$a_m = \frac{a_{m-1}}{(m-1/2)(2m-1-1)-1} = \frac{a_{m-1}}{(2m-1)(m-1)-1} = \frac{a_{m-1}}{m(2m-3)}$$

$$a_1 = \frac{a_0}{(-1)}\ ;\ a_2 = \frac{a_1}{2.1} = -\frac{a_0}{2},\ a_3 = \frac{a_2}{3.3} = -\frac{a_0}{18}$$

$$y_1(x) = x^{-1/2}\left\{1 - x - \frac{1}{2}x^2 - \frac{x^3}{18} - \cdots - \frac{x^m}{m!\,1.3.5.7..(2m-3)} + \cdots\right\}$$

$r = +1$: $$a_m = \frac{a_{m-1}}{(m+1)(2m+1)-1} = \frac{a_{m-1}}{m(2m+3)}$$

$$a_1 = \frac{a_0}{1.5},\ a_2 = \frac{a_1}{2.7} = \frac{a_0}{(1.2)(5.7)},\ a_3 = \frac{a_2}{3.9} = \frac{a_0}{(1.2.3)(5.7.9)}$$

$$y_2(x) = x\left\{1 + \frac{x}{1.5} + \frac{x^2}{2!\,5.7} + \cdots + \frac{x^m}{m!\,5.7..(2m+3)} + \cdots\right\}$$

RADIUS OF CONVERGENCE

$y_1(x)$: $$\left|\frac{a_m}{a_{m-1}}\right||x| = \frac{|x|}{m(2m-3)} \xrightarrow{m\to\infty} 0 < 1 \Rightarrow \rho = \infty$$

SIMILARLY WITH THE SERIES FOR $y_2(x)$