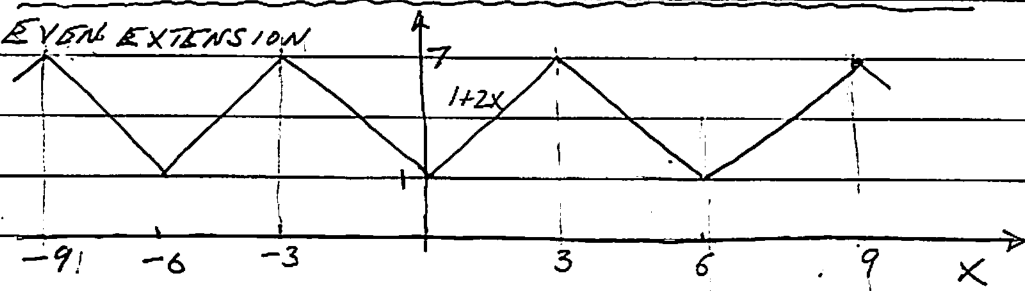
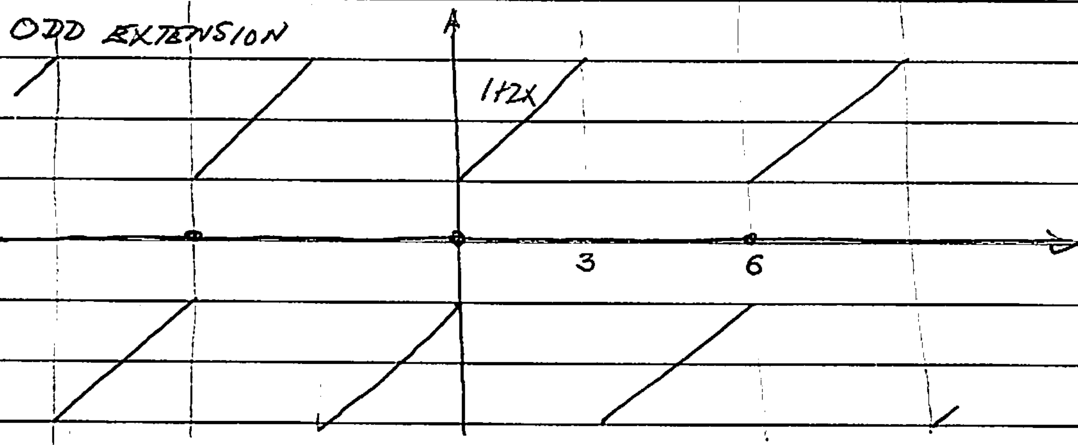
MATH 257/316 MIDTERM 1 SECTION 101 OCT 2008

1. (a) EVEN EXTENSION



ODD EXTENSION



(b) USE EVEN EXTENSION $\quad f_{EVEN}(-x) = f_{EVEN}(x) \qquad L=3.$

$$f_{EVEN}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right)$$

$$a_0 = \frac{2}{3}\int_0^3 (1+2x)\,dx = \frac{2}{3}\left[x+x^2\right]_0^3 = 2+6 = 8$$

$$a_n = \frac{2}{3}\int_0^3 (1+2x)\cos\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\left[\cancel{\left(\frac{3}{n\pi}\right)\sin\left(\frac{n\pi x}{3}\right)\Big]_0^3} + 2\begin{cases} 0 & n \text{ EVEN} \\ -\dfrac{18}{n^2\pi^2} & n \text{ ODD}\end{cases}\right]$$

$$= \begin{cases} 0 & n \text{ EVEN} \\ -\dfrac{24}{n^2\pi^2} & n \text{ ODD.}\end{cases} = -\frac{24}{(2k+1)^2\pi^2} \qquad k=0,1,\dots$$

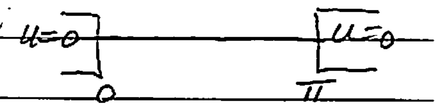
$$\therefore\ 1+2x = \frac{8}{2} - \frac{24}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos\left(\frac{(2k+1)\pi x}{3}\right)}{(2k+1)^2}$$

(c)

SET $x=0$: $\quad 1 = 4 - \dfrac{24}{\pi^2}\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$

$$\therefore\ \frac{\pi^2}{8} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

2. $u_t = u_{xx}$ $\quad 0 < x < \pi$



$u(0,t) = 0 = u(\pi,t)$

$u(x,0) = 2\cos x \sin x = \sin 2x$

LET $u(x,t) = X(x)\,T(t)$

$X\dot{T}(t) = X''T(t)$

$\dfrac{\dot{T}(t)}{T(t)} = \dfrac{X''}{X(x)} = -\lambda^2$ CONST

$\boxed{T(t)}$ $\quad \dot{T} = -\lambda^2 T \Rightarrow T(t) = Ce^{-\lambda^2 t}$

$\boxed{X(x)}$ $\quad \left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(\pi) \end{array}\right\}$ $\quad X(x) = A\cos\lambda x + B\sin\lambda x$

$X(0) = A = 0$

$X(\pi) = B\sin(\lambda\pi) = 0 \Rightarrow \lambda_n = n \quad n = 1, 2, \ldots$

$\therefore$ EIGENVALUES ARE $\lambda_n = n$ $\; n = 1, 2, \ldots$ WITH EIGENFUNCTIONS $X_n = \sin(nx)$

THE GENERAL SOLUTION TO THE BVP IS THUS

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin(nx)$$

NOW TO MATCH THE INITIAL CONDITION:

$$\sin 2x = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

BY INSPECTION $b_n = \begin{cases} 0 & n \neq 2 \\ 1 & n = 2 \end{cases} = \delta_{n2}$

ALTERNATIVELY BY DIRECT CALCULATION

$b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin(2x)\sin(nx)\,dx = \dfrac{2}{\pi}\begin{cases} 0 & n \neq 2 \\ \pi/2 & n = 2 \end{cases}$ BY ORTHONORMALITY

$\therefore \; u(x,t) = e^{-4t}\sin 2x$

3. $Ly = 2xy'' + (1+x)y' + y = 0$ $\quad P(x) = 2x \quad Q(x) = (1+x) \quad R(x) = 1$

(a) SINCE $P(x) = 2X > 0$ FOR ALL $X > 0$, ALL $X > 0$ ARE ORDINARY POINTS

SINCE $P(0) = 0$ $\;X = 0$ IS A SINGULAR POINT

$$\lim_{X\to 0} x\frac{Q(x)}{P(x)} = \lim_{X\to 0} \frac{x(1+x)}{2x} = \frac{1}{2} < \infty \; ; \; \lim_{X\to 0} x^2\frac{R(x)}{P(x)} = \lim_{X\to 0} \frac{x^2 \cdot 1}{2x} = 0 < \infty$$

THUS $X = 0$ IS A REGULAR SINGULAR POINT.

THE INDICIAL EQ IN THIS CASE IS: $r(r-1) + \frac{r}{2} = r\left(r - \frac{1}{2}\right) = 0$

THUS THE EXPONENTS ARE $r = 0$ & $r = 1/2$.

(b) SINCE $X = 0$ IS A RSP WE ASSUME A FROBENIUS EXP OF THE FORM:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \overset{m-1=n}{=} \sum_{m=1}^{\infty} a_{m-1} x^{m+r-1}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \overset{m=n}{=} \sum_{m=1}^{\infty} a_m (m+r) x^{m+r-1} + a_0 r x^{r-1}$$

$$xy' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} \overset{m-1=n}{=} \sum_{m=1}^{\infty} a_{m-1} (m-1+r) x^{m+r-1}$$

$$xy'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} \overset{m=n}{=} \sum_{m=1}^{\infty} a_m (m+r)(m+r-1) x^{m+r-1} + a_0 r(r-1) x^{r-1}$$

IN THESE TWO SERIES LET $m-1 = n \Rightarrow m = n+1 \quad n = 0 \Rightarrow m = 1$

$$0 = Ly = \sum_{m=1}^{\infty} \left[2a_m(m+r)(m+r-1) + a_m(m+r) + a_{m-1}(m+r-1) + a_{m-1}\right] x^{m+r-1} + a_0\left[2r(r-1) + r\right] x^{r-1}$$

$x^{r-1} \Rightarrow \; a_0[2r^2 - 2r + r] = a_0 r(2r-1) = 0 \quad r = 0,\; r = 1/2$ ARE THE ROOTS

$x^{m+r-1} \Rightarrow \; a_m(m+r)[2(m+r-1)+1] + a_{m-1}[(m+r-1)+1] = 0 \quad m \geq 1$

THE RECURSION IS: $\quad a_m = -\dfrac{a_{m-1}}{2m+2r-1}$ $\quad$ since $(m+r) \neq 0$ FOR $m \geq 1$

$r = 0$: $\quad a_m = -\dfrac{a_{m-1}}{2m-1}$; $\; a_1 = -\dfrac{a_0}{1}$, $a_2 = -\dfrac{a_1}{3} = +\dfrac{a_0}{3}$, $a_3 = -\dfrac{a_2}{5} = -\dfrac{a_0}{5.3}$ ...

$$\therefore \; y_1(x) = a_0\left\{1 - x + \frac{x^2}{3} - \frac{x^3}{5.3} + \dots\right\}$$

$r = 1/2$: $\quad a_m = -\dfrac{a_{m-1}}{2m}$; $\; a_1 = -\dfrac{a_0}{2}$, $a_2 = -\dfrac{a_1}{4} = +\dfrac{a_0}{4.2}$; $a_3 = -\dfrac{a_2}{6} = -\dfrac{a_0}{6.4.2}$

$$y_2(x) = a_0 x^{1/2}\left[1 - \frac{x}{2} + \frac{x^2}{4.2} - \frac{x^3}{6.4.2} + \dots\right]$$