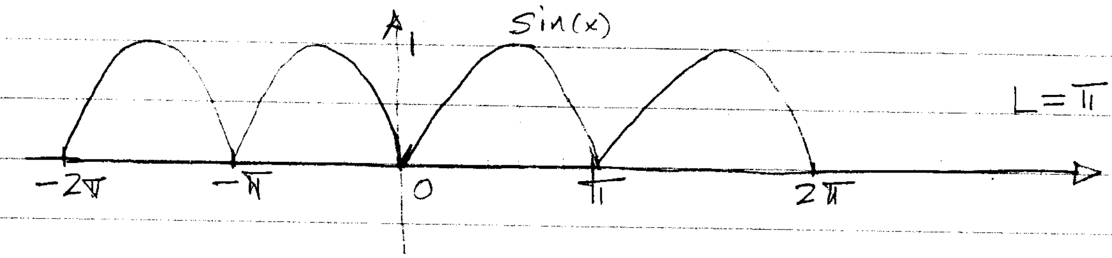
MATH 257/316 MIDTERM 1 SECTION 101 SOLUTIONS 17OCT'07

1. $f(x) = \sin x \quad 0<x<\pi$

(a)



(b)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin(x)\,dx = -\frac{2}{\pi}\cos x\Big|_0^\pi = \frac{2}{\pi}[1-(-1)] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x \cos(nx)\,dx = \frac{2}{\pi}\left\{\frac{-2}{n^2-1}\right\} = \frac{4}{\pi}\,\frac{1}{1-n^2} \quad n \text{ EVEN}$$

$$\therefore\ f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{1-(2n)^2}$$

2. $u_t = u_{xx} \quad 0<x<\pi \quad t>0$

BC: $u(0,t) = 0 = u_x(\pi,t)$

IC: $u(x,0) = 2\sin(3x/2)$

LET $u(x,t) = X(x)T(t) \Rightarrow \dfrac{\dot{T}(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda^2$ CONST

$\dot{T} = -\lambda^2 T \Rightarrow T(t) = Ce^{-\lambda^2 t}$

$X'' + \lambda^2 X = 0 \Rightarrow X(x) = A\cos\lambda x + B\sin\lambda x, \quad X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$X(0)=0 \Rightarrow A=0; \quad X'(\pi) = B\lambda\cos\lambda\pi = 0 \Rightarrow \boxed{\lambda_n = (2n+1)/2 \quad n=0,1,2..}$

THE CORRESPONDING EIGENFUNCTIONS ARE $\boxed{X_n = \sin((2n+1)x/2)}$

$$\therefore\ u(x,t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{(2n+1)x}{2}\right)e^{-\left(\frac{2n+1}{2}\right)^2 t}$$

APPLY INITIAL CONDITION

$$2\sin(3x/2) = u(x,0) = \sum_{n=0}^{\infty} B_n\sin\left(\frac{2n+1}{2}x\right) = B_0\sin\left(\frac{x}{2}\right) + B_1\sin\left(\frac{3x}{2}\right) + B_2\sin\left(\frac{5x}{2}\right) + \dots$$

MATCHING COEFFICIENTS $B_1 = 2$ AND $B_n = 0$ IF $n \neq 1$

OR BY DIRECT CALCULATION $\displaystyle\int_0^\pi \sin(3x/2)\sin\left(\frac{2n+1}{2}x\right)dx = \begin{cases} 0 & n\neq 1 \\ \pi/2 & n=1\end{cases}$

WHICH YIELDS THE SAME RESULT.

$$\therefore\ u(x,t) = 2e^{-\left(\frac{3}{2}\right)^2 t}\sin\left(\frac{3}{2}x\right) \quad \text{IS THE SOLUTION.}$$

$$Ly = 4x^2y'' + xy' - (1-3x)y = 0$$

3. (a) $x > 0$ ARE ALL ORDINARY POINTS

$x = 0$ IS A REGULAR SINGULAR POINT SINCE

$$\lim_{x\to 0} x\left(\frac{x}{4x^2}\right) = \frac{1}{4} = p_0 < \infty \qquad \lim_{x\to 0} x^2\left(\frac{1-3x}{4x^2}\right) = \frac{1}{4} = q_0 < \infty.$$

(b) LET $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{m=1}^{\infty} a_m x^{m+r} + a_0 x^r$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+r+1} = \sum_{m=1}^{\infty} a_{m-1} x^{m+r}$$

$$xy' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} = \sum_{m=1}^{\infty} a_m (m+r) x^{m+r} + a_0 r x^r$$

$$x^2y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} = \sum_{m=1}^{\infty} a_m (m+r)(m+r-1) x^{m+r} + a_0 r(r-1) x^r$$

$m + \cancel{r} = n + \cancel{r} + 1 \Rightarrow m = n+1 \qquad n = 0 \Rightarrow m = 1$

$$Ly = \sum_{m=1}^{\infty} \left\{ a_m \left[4(m+r)(m+r-1) + (m+r) - 1 \right] + 3a_{m-1} \right\} x^{m+r} + a_0 \left[4r(r-1) + r - 1 \right] x^r = 0$$

$\boxed{x^r}$ $\quad a_0[4r^2 - 3r - 1] = a_0(4r+1)(r-1) = 0 \qquad r = 1 \quad r = -1/4.$

$\boxed{x^{m+r} \; m \geq 1}$ $\quad a_m = \dfrac{-3a_{m-1}}{(m+r)[4(m+r-1)+1] - 1}$

$r = 1$: $\quad a_m = \dfrac{-3a_{m-1}}{(m+1)(4m+1) - 1}$

$$a_1 = \frac{-3a_0}{2(5)-1} = \frac{-3a_0}{9} = \frac{-a_0}{3}\ ; \quad a_2 = \frac{-3a_1}{3(9)-1} = \frac{-3a_1}{26} = +\frac{a_0}{26}$$

$$\therefore \; y_1(x) = a_0 x \left[1 - \frac{x}{3} + \frac{x^2}{26} - \dots \right]$$

$r = -1/4$: $\quad a_m = \dfrac{-3a_{m-1}}{(m-1/4)(4m - \cancel{1} - 4 + \cancel{1}) - 1} = \dfrac{-3a_{m-1}}{(4m-1)(m-1) - 1}$

$$a_1 = \frac{-3a_0}{3.0 - 1} = 3a_0\ ; \quad a_2 = \frac{-3a_1}{7.1 - 1} = \frac{-3}{6} a_1 = -\frac{3}{2} a_0$$

$$\therefore \; y_2(x) = a_0 x^{-1/4} \left[1 + 3x - \frac{3}{2} x^2 + \dots \right]$$