MATH 257/316 MIDTERM 1 SECTION 101. 11 OCT 2006

$$u_t = \alpha^2 u_{xx} \qquad 0<x<L,\ t>0$$

$$BC: \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$$

$$IC: \quad u(x,0) = f(x)$$

LET $\quad u(x,t) = X(x)T(t)$

$$X(x)\dot{T}(t) = \alpha^2 X''(x)T(t)$$

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 = \text{CONST}$$

$$\dot{T}(t) = -\lambda^2\alpha^2 T(t) \Rightarrow \quad T(t) = Ce^{-\lambda^2\alpha^2 t}$$

$$X''(x) + \lambda^2 X(x) = 0 \quad \Rightarrow \quad X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

$$\frac{\partial u}{\partial x}(0,t) = X'(0)T(t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = X'(L)T(t) = 0$$

FOR A NONTRIVIAL SOL'N $T(t) \neq 0 \quad X'(0) = 0 = X'(L)$

$$X'(x) = -A\lambda\sin(\lambda x) + B\lambda\cos(\lambda x)$$

$$X'(0) = B\lambda = 0 \Rightarrow B = 0 \quad \text{OR } \lambda = 0$$

$$X'(L) = -A\lambda\sin(\lambda L) = 0 \Rightarrow \lambda L = n\pi \qquad n = 0,1,\ldots$$

EIGENVALUES ARE $\quad \lambda_n = (n\pi/L) \quad n = 0,1,\ldots$ AND CORRESPONDING EIGENFUNCTIONS $\quad X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

BY SUPERPOSITION THE GENERAL SOLUTION IS OF THE FORM

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

IMPOSE IC: $\quad f(x) = u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$

WHERE USING $\displaystyle\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L/2 & m=n\neq 0 \\ 0 & m \neq n \\ L & m=n=0 \end{cases}$

WE OBTAIN $\quad A_n = \frac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ AND $A_0 = \frac{1}{L}\displaystyle\int_0^L f(x)\,dx$

FOR $L=1$, $f(x) = x$: $\quad A_0 = \frac{1}{2}$

$$A_n = 2\int_0^1 x\cos(n\pi x)\,dx = 2\left[x\,\cancel{\frac{\sin(n\pi x)}{n\pi}}\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\,dx\right]$$

$$= \frac{2}{(n\pi)^2}\left[\cos(n\pi x)\right]_0^1 = \frac{2}{(n\pi)^2}\left[(-1)^n - 1\right] = \frac{-4}{[(2m+1)\pi]^2} \qquad m = 0,1,\ldots$$

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{m=0}^{\infty} e^{-\alpha^2[(2m+1)\pi]^2 t}\,\frac{\cos((2m+1)\pi x)}{(2m+1)^2}$$

2. $Ly = 2x^2 y'' - 3x y' + (3+x) y = 0 \quad (1)$

(a) ALL $x_0 > 0$ ARE ORDINARY POINTS FOR (1)

$x = 0$ IS A REGULAR SINGULAR POINT SINCE

$$y'' - \frac{3x}{2x^2} y' + \frac{(3+x)}{2x^2} y = 0$$

SINCE $x\left(\frac{-3x}{2x^2}\right) \xrightarrow{x\to 0} -\frac{3}{2}$ $\quad x^2\left(\frac{3+x}{2x^2}\right) \xrightarrow{x \to 0} 3$ ARE FINITE $x=0$ IS A REGULAR SINGULAR POINT.

(b) LET $t = 1/x$ $\quad \frac{d}{dx} = \left(\frac{d}{dt}\cdot\right)\left(\frac{dt}{dx}\right) = -t^2 \frac{d}{dt}$.

$$\frac{d^2}{dx^2} = \left(-t^2\frac{d}{dt}\right)\left(-t^2\frac{d}{dt}\right) = t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt}$$

$$0 = Ly = \frac{2}{t^2}\left\{t^4\ddot{y} + 2t^3\dot{y}\right\} - \frac{3}{t}\left(-t^2\dot{y}\right) + \left(3 + \frac{1}{t}\right)y$$

$$= 2t^2\ddot{y} + 4t\dot{y} + 3t\dot{y} + \left(3 + \frac{1}{t}\right)y$$

$\div 2t^2$

$$\ddot{y} + \frac{7}{2t}\dot{y} + \left(\frac{3}{2t^2} + \frac{1}{2t^3}\right)y = 0$$

THIS TERM IS A PROBLEM

$t^2\left(\frac{1}{2t^3}\right) \xrightarrow{t \to 0} \infty$ SO THAT $t=0$ IS AN IRREGULAR POIN

$\therefore$ THE POINT AT ∞ IS AN IRREGULAR SINGULAR POINT.

(c) REWRITE (1) IN THE FORM: $2[(x-1)^2 + 2(x-1) + 1]y'' - 3[(x-1)+1]y' + [(x-1)+4]y = 0$

LET $t = (x-1)$ $\quad \frac{d}{dt} = \frac{d}{dx}$: $2[t^2 + 2t + 1]y'' - 3(t+1)y' + (t+4)y = 0 \quad (2)$

SINCE $t=1$ IS AN ORDINARY POINT WE ASSUME THAT

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

SUBSTITUTE THIS SERIES INTO (2) AND COLLECT LIKE POWERS IN t AND EQUATE THE COEFFICIENTS OF THESE TERMS TO ZERO. THESE CONDITIONS YIELD RECURSIONS BETWEEN THE a_n WHICH YIELD TWO INDEPENDENT SOLUTIONS.

(d) $x=0$ IS A RSP FOR $Ly = 2x^2y'' - 3xy' + (3+x)y = 0$.
SO WE ASSUME A SOLUTION OF THE FORM $y = \sum_{n=0}^{\infty} a_n x^{n+r}$.

$$0 = Ly = \sum_{n=0}^{\infty} a_n\left[2(n+r)(n+r-1) - 3(n+r) + 3\right]x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1}$$

$n=0 \Rightarrow m=1$
$m=n+1 \Rightarrow n=m-1$

$$0 = a_0\{2r(r-1) - 3r + 3\}x^r + \sum_{m=1}^{\infty}\{a_m[(m+r)(2(m+r-1)-3)+3] + a_{m-1}\}x^{m+r}$$

$x^r >$ $a_0\{2r^2 - 5r + 3\} = a_0(2r-3)(r-1) = 0$ $\quad r=1,\ r=3/2$ 2 DISTINCT ROOTS

$x^{m+r}, m \geq 1 >$ $$a_m = \frac{-a_{m-1}}{[(m+r)(2(m+r)-5)+3]}$$

$r=1$: $$a_m = \frac{-a_{m-1}}{(m+1)(2m-3)+3} = \frac{-a_{m-1}}{m(2m-1)}$$

$$a_1 = \frac{-a_0}{1.1};\quad a_2 = \frac{-a_1}{2.3} = \frac{(-1)^2 a_0}{2.3}\quad a_3 = \frac{-a_2}{3(5)} = \frac{(-1)^3 a_0}{3!\,3.5},$$

$$a_m = \frac{(-1)^m a_0}{m!\,3.5..(2m-1)}$$

$$y_1(x) = x\left\{1 - \frac{x}{1} + \frac{x^2}{2!3} + \dots + \frac{(-1)^m x^m}{m!\,3.5..(2m-1)} + \cdots\right\}$$

$r=\frac{3}{2}$: $$a_m = \frac{-a_{m-1}}{[(m+\frac{3}{2})(2(m+3/2)-5)+3]} = \frac{-a_{m-1}}{m(2m+1)}$$

$$a_1 = \frac{-a_0}{1.3},\quad a_2 = \frac{-a_1}{2.5} = \frac{(-1)^2 a_2}{(2.1)(5.3)}\quad a_3 = \frac{-a_2}{3.7} = \frac{(-1)^3 a_0}{3!\,3.5.7}$$

$$a_m = \frac{(-1)^m a_0}{m!\,3.5.7..(2m+1)} = \frac{(-1)^m a_0\, 2^m \cancel{m!}}{\cancel{m!}(2m+1)!}$$

$$y_2(x) = x^{3/2}\left\{1 - \frac{x}{3} + \frac{x^2}{30} - \dots + \frac{(-1)^m (2x)^m}{(2m+1)!} + \cdots\right\}$$

$$= x^{3/2}\sum_{m=0}^{\infty}\frac{(-1)^m (2x)^m}{(2m+1)!}$$

RADIUS OF CONVERGENCE $\left|\frac{a_m}{a_{m-1}}\right||x| = \frac{|x|}{m(2m-1)} \xrightarrow{m\to\infty} 0 < 1,\ \rho = \infty$.

SIMILARLY FOR $y_2(x)$.