MATH 257/316 MIDTERM 1 SECTIONS 101/102 2009

Q1. $Ly = 6x^2y'' + xy' + (1+x)y = 0$

(a) • $x=0$ IS A SINGULAR POINT SINCE $6x^2$ VANISHES

TO SEE IF IT IS A RSP: $\lim_{x\to 0} x\left(\frac{x}{6x^2}\right) = \frac{1}{6} = p_0 < \infty$, $\lim_{x\to 0} x^2\frac{(1+x)}{6x^2} = \frac{1}{6} = q_0 < \infty$

SINCE $|q_0| < \infty$ & $|p_0| < \infty$ $x=0$ IS A RSP.

• THE POINTS $0 < x < \infty$ ARE ALL ORDINARY POINTS

• CONSIDER THE POINT AT ∞: LET $z = 1/x$ THEN AS $x \to \infty$ $z \to 0$.

$$\frac{d}{dx} = \frac{d}{dz}\frac{dz}{dx} = -\frac{1}{x^2}\frac{d}{dz} = -z^2\frac{d}{dz}, \quad \frac{d^2}{dz^2} = \left(-z^2\frac{d}{dz}\right)\left(-z^2\frac{d}{dz}\right) = z^4\frac{d^2}{dz^2} + 2z^3\frac{d}{dz}$$

$$\therefore\ 6x^2y'' + xy' + (1+x)y = \frac{6}{z^2}\left\{z^4\frac{d^2}{dz^2} + 2z^3\frac{d}{dz}\right\}y + \left(\frac{1}{z}\right)\left(-z^2\frac{d}{dz}\right)y + \left(1+\frac{1}{z}\right)y = 0$$

$$\therefore\ 6z^2\ddot{y} + 11z\dot{y} + (1 + 1/z)y = 0 \qquad \text{WHERE } \frac{d}{dz}y = \dot{y}$$

$z=0$ (OR $x\to\infty$) IS A SINGULAR POINT TO CHECK IF IT IS A RSP

$\lim_{z\to 0} z\left(\frac{11z}{6z^2}\right) = \frac{11}{6}$ $\quad \lim_{z\to 0} z^2\frac{(1+1/z)}{6z^2} \to \infty \Rightarrow z=0$ $x\to\infty$ IS AN IRREGULAR SP.

(b) • SINCE $x_0=1$ IS AN ORDINARY POINT WE NEED TO ASSUME $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$

• THE NEAREST SINGULAR POINT TO $x_0=1$ IS $x=0$ SO THE MINIMUM RADIUS OF CONVERGENCE IS $\rho = 1$.

(c) SINCE $X=0$ IS A RSP WITH $p_0 = 1/6 = q_0$ THE INDICIAL EQUATION IS

$$r(r-1) + \frac{1}{6} r + \frac{1}{6} = 0 \Rightarrow 6r^2 - 5r + 1 = (3r-1)(2r-1) = 0 \quad r = \tfrac{1}{2}, \tfrac{1}{3}.$$

ASSUME $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ $\quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$ $\quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$0 = Ly = \sum_{n=0}^{\infty} \underbrace{a_n [6(n+r)(n+r-1) + (n+r) + 1] x^{n+r}}_{m=n} + \underbrace{a_n x^{n+r+1}}_{m=n+1 \quad n=m-1}$$

$n=0 \Rightarrow m=1$

$$\therefore 0 = \sum_{m=1}^{\infty} \{a_m [6(m+r)(m+r-1) + (m+r) + 1] + a_{m-1}\} x^{m+r} + a_0 [6r(r-1) + r + 1] x^r$$

EQUATE COEFFICIENTS TO ZERO

$x^r >$ $\quad 6r^2 - 5r + 1 = (3r-1)(2r-1) = 0 \quad a_0$ ARBITRARY

$x^{m+r} >$ $$a_m = \frac{-a_{m-1}}{6(m+r)(m+r-1) + m + r + 1}$$

$\underline{r = 1/2}$: $$a_m = \frac{-a_{m-1}}{6(m+\frac{1}{2})(m-\frac{1}{2}) + m + 3/2} = \frac{-a_{m-1}}{6(m^2 - 1/4) + m + 3/2} = \frac{-a_{m-1}}{(6m+1)m}$$

$$a_1 = \frac{-a_0}{7.1} \qquad a_2 = \frac{-a_1}{13.2} = \frac{+a_0}{13.7.2.1}$$

$$\therefore y_0(x) = x^{1/2} \left[1 - \frac{x}{7} + \frac{x^2}{13 \cdot 14} - \ldots\right] \quad \text{IS ONE SOL'N}$$

$\underline{r = 1/3}$: $$a_m = \frac{-a_{m-1}}{6(m+\frac{1}{3})(m-\frac{2}{3}) + m + 4/3} = \frac{-a_{m-1}}{2(3m+1)(m-\frac{2}{3}) + m + 4/3} = \frac{-a_{m-1}}{(6m-1)m}$$

$$a_1 = \frac{-a_0}{5.1} \qquad a_2 = \frac{-a_1}{11.2} = \frac{+a_0}{110}$$

$$\therefore y_1(x) = x^{1/3} \left[1 - \frac{x}{5} + \frac{x^2}{110} - \ldots\right]$$

THE GENERAL SOL'N IS THUS OF THE FORM

$$y(x) = A\, y_0(x) + B\, y_1(x).$$

Q2: $u_t = \alpha^2 u_{xx} \qquad 0 < x < \pi, \quad t > 0$

$u_x(0,t) = 0 = u_x(\pi, t)$

$u(x,0) = 1$

SOL'N: LET $u(x,t) = X(x)\,T(t)$

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \text{CONST} = -\lambda^2 \quad \text{(FOR NONTRIVIAL SOL'NS)}$$

$$T(t) = C e^{-\alpha^2 \lambda^2 t}$$

$$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X'(\pi) \end{array}\right\} \quad \begin{array}{l} X(x) = A\cos\lambda x + B\sin\lambda x \\ X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x \end{array}$$

$\lambda \neq 0$: $X'(0) = B\lambda = 0 \Rightarrow B = 0$ SINCE $\lambda \neq 0$

$X'(\pi) = -A\lambda\sin(\lambda\pi) = 0 \Rightarrow \lambda\pi = n\pi \quad n = 1, 2, \ldots$

WE HAVE EIGENVALUES $\lambda_n = n,\ n = 1, 2 \ldots$

AND EIGENFUNCTIONS $X_n(x) = \cos(nx)$.

$\lambda = 0$: $X'' = 0 \qquad X(x) = Bx + A \qquad X' = B$

$X'(0) = B = 0 \qquad X'(\pi) = 0$ IS SATISFIED AUTOMATICALLY

$\therefore \quad X(x) = A \cdot 1$

$\therefore \quad \lambda_0 = 0$ IS AN EIGENVALUE AND $X_0(x) = 1$ IS THE CORRESPONDING EIGENFUNCTION

THE GENERAL SOLUTION IS THUS

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\alpha^2 n^2 t} \cos(nx)$$

TO MATCH THE INITIAL CONDITION

$$1 = u(x,0) = A_0 1 + \sum_{n=1}^{\infty} A_n \cos(nx)$$

TO FIND THE A_n WE PROJECT 1 ONTO THE $\cos(mx)$

METHOD 1: $0 = (A_0 - 1)\cdot 1 + \sum_{n=1}^{\infty} A_n \cos(nx)$ AND SINCE THE FUNCTIONS $\{1, \cos(mx)\}$ ARE INDEPENDENT $A_0 = 1,\ A_n = 0\ \ n \geq 1$

METHOD 2: $m \neq 0$ $\displaystyle\int_0^\pi 1\cos(mx)\,dx = \frac{\sin(mx)}{m}\Big|_0^\pi = A_0 \cancelto{0}{\int_0^\pi \cos(mx)\,dx} + \sum_{n=1}^{\infty} A_n \int_0^\pi \cos(mx)\cos(nx)\,dx$

$$0 = A_m (\pi/2) \Rightarrow A_m = 0 \quad m \geq 1$$

$m = 0$: $\displaystyle\int_0^\pi 1\,dx = A_0 \int_0^\pi 1\,dx + \sum_{n=1}^{\infty} A_n \cancelto{0}{\int_0^\pi 1\cdot\cos(nx)\,dx}$

$$A_0 = 1$$

$$\therefore \quad u(x,t) = 1$$