Practise Problems

$\boxed{\text{Problem 1 and 2:}}$ Consider the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0<x<L,\ t>0 \\ IC:\ u(0,t) = 0 = u(L,t) & \\ BC:\ u(0,t) = f(x),\ u_t(0,t) = g(x) & \end{cases}$$

Then the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(b_n \cos\left(c \frac{n\pi}{L} t\right) + c_n \sin\left(c \frac{n\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$\frac{cn\pi}{L} c_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

Hence, in Problem 1, we obtain

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos(nt) \sin(n\pi x)$$

At time $t=0$:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \stackrel{!}{=} f(x)$$

with $b_n = 2\int_0^1 f(x) \sin(n\pi x)\, dx$

Let's compute b_n: Using integration by parts,

$$\underline{b_n} = \frac{3}{5} \int_0^{1/3} x \sin(n\pi x)\, dx + \frac{3}{10} \int_{1/3}^{1} (1-x) \sin(n\pi x)\, dx$$

$$= \frac{9}{10\pi^2} \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2}$$

Hence, the solution is

$$u(x,t) = \frac{9}{10\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2} \sin(n\pi x)\cos(nt)$$

Similarly, we obtain for Problem 2:

$$u(x,t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(4n^2-1)^2} \sin(2nx)\sin(2nt)$$

$\boxed{\text{Problem 3:}}$ $u(x,y) = X(x)Y(y)$

homogeneous in x-direction

$$\Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\mu^2 \quad , \quad \mu \geq 0$$

$\boxed{\text{in } x\text{-direction}}$
$$\begin{cases} X'' + \mu^2 X = 0, \quad 0 < x < \pi \\ X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$$

The eigenvalues and eigenfunctions are: $\underline{\underline{\mu_n^2 = n^2}}$ $\quad n = 0, 1, 2, \ldots$

$$\underline{\underline{X_n(x) = \cos(nx)}}$$

($n = 0$ corresponds to the zero eigenvalue with the constant eigenfunction $X_0 = 1$)

$\boxed{\text{in } y\text{-direction}}$ $\quad Y'' - \mu_n^2 Y = 0, \quad 0 < y < \pi$

$$Y'(\pi) = 0$$
$$Y(0) = ?$$

$\underline{n = 0:}$ $\quad Y = Ay + B$: $Y'(\pi) = 0 \Rightarrow A = 0$

$$\underline{\underline{Y_0 = \text{const} = B}}$$

$\underline{n > 0:}$ $\quad$ Since $Y'(\pi) = 0$, we have

$$\underline{\underline{Y_n = B\cosh(n(y-\pi))}}$$

Then $Y_n'(\pi) = Bn \sinh(n(y-\pi))\Big|_{y=\pi} = 0$

The general solution is

$$\boxed{u(x,y) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cosh(n(y-\pi)) \cos(nx)}$$

$$y=0: \quad u(x,0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cosh(n\pi) \cos(nx) \overset{!}{=} f(x)$$

$$\Rightarrow \quad B_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$B_n \cosh(n\pi) = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$$

Let's compute these integrals:

- $$\frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi/2} 1\, dx = \underline{\underline{1}}$$

- $$\frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx = \frac{2}{\pi} \int_0^{\pi/2} \cos(nx)\, dx = \frac{2}{\pi} \left. \frac{\sin(nx)}{n} \right|_{\underbrace{0}_{=0}}^{\pi/2} = \frac{2}{\pi} \frac{\sin\left(\frac{\pi n}{2}\right)}{n}$$

$$= \frac{2}{\pi n} \times \begin{cases} 0 & n = 2, 4, 6, 7, 8, \dots \\ 1 & n = 1, 5, 9, \dots \\ -1 & n = 3, 7, 11 \end{cases} \qquad n = 2k+1, \ k = 0, 1, \dots$$

$$\Rightarrow \quad \boxed{u(x,y) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2\,(-1)^k}{\pi(2k+1)\cosh((2k+1)\pi)} \cos((2k+1)x)}$$

Problem 4

We first look for a function $v(x)$ that satisfies the boundary conditions as well as $v''(x) = 0$:

$$v(x) = 2x + 2$$

$$\Rightarrow \quad v''(x) = 0$$
$$v(0) = 2$$
$$v(1) = 4$$

Then set $u(x,y) = v(x) + w(x,y)$

$$0 = u_{xx} + u_{yy} = \underbrace{\cancel{v''}}_{=0} + w_{xx} + w_{yy} \Rightarrow w_{xx} + w_{yy} = 0$$

$$2 = u(0,y) = \overbrace{\cancel{v(0)}}^{=2} + w(0,y) \quad \Rightarrow \quad w(0,y) = 0$$

$$4 = u(1,y) = \overbrace{\cancel{v(1)}}^{=4} + w(1,y) \quad \Rightarrow \quad w(1,y) = 0$$

$$f(x) = u(x,0) = v(x) + w(x,0) \quad \Rightarrow \quad w(x,0) = f(x) - v(x)$$

The problem for w is homogeneous and can be solved as in class. We obtain

$$\boxed{u(x,y) = 2x + 2 + \sum_{n=1}^{\infty} a_n e^{-n\pi y} \sin(n\pi x)}$$

where

$$\boxed{a_n = 2 \int_0^1 (f(x) - v(x)) \sin(n\pi x)}$$

$\boxed{\text{Problem 5:}}$ Define a linear function that satisfies the boundary conditions: $\underline{\underline{v(\theta) = \frac{2\theta}{\alpha} - 1}}$: $v(0) = -1$ & $v(\alpha) = 1$

Then set $u(r,\theta) = v(\theta) + w(r,\theta)$.

Since $v''(\theta) = 0$, we obtain that w satisfies

$$\begin{cases} w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} = 0, \; 0 < r < a, \; 0 < \theta < \alpha \\ w(r,0) = 0 \;\; \& \;\; w(r,\alpha) = 0 \\ w(a,\theta) = f(\theta) - v(\theta) \end{cases}$$

This problem is homogeneous and has been solved in class.

We obtain

$$\boxed{u(r,\theta) = \overbrace{\frac{2\theta}{\alpha} - 1}^{= v(\theta)} + \sum_{n=1}^{\infty} c_n r^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi\theta}{\alpha}\right)}$$

$$\underline{\underline{\text{where } c_n = \frac{2}{\alpha} a^{-\left(\frac{n\pi}{\alpha}\right)} \int_0^{\alpha} \left(f(\theta) - v(\theta)\right) \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta}}$$

Problem 6

$$u(r,\theta) = R(r)\,\Theta(\theta)$$

periodic boundary conditions in θ

Separation yields $\dfrac{r^2\left(R'' + \frac{1}{r}R'\right)}{R} = -\dfrac{\Theta''}{\Theta} = +\mu^2, \quad \mu \geq 0$

$\boxed{\text{in } \theta:}$ $\quad \Theta'' + \mu^2\Theta = 0, \quad 0 < \theta < 2\pi$

$\Theta(\theta) = \Theta(\theta + 2\pi)$

as in class, the eigenvalues and eigenfunctions are

$$\left.\begin{aligned} \mu_n^2 &= n^2, \quad n = 0, 1, 2, 3, \ldots \\ \Theta_n &= A\cos(n\theta) + B\sin(n\theta), \quad n = 0, 1, 2, \end{aligned}\right\} \text{ n=0 corresponds to zero eigenvalue and constant eigenfunction}$$

$\boxed{\text{in } r:}$ $\quad r^2R'' + rR' - n^2R = 0$

$\boxed{n=0}$ $\quad \underline{\underline{R_0 = B_0 \ln(r) + A_0}}$

boundedness for $r \to \infty$: $\underline{\underline{B_0 = 0}}$ $\quad \Big\}\ \underline{\underline{R_0 = \text{const}}}$

$\boxed{n > 1}$ $\quad R = Br^{-n} + Dr^{+n}$

boundedness for $r \to \infty$: $\underline{\underline{D = 0}}$ $\quad \Big\}\ \underline{\underline{R_n = Br^{-n}}}$

$\Rightarrow$ The general solution is

$$\boxed{u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left\{A_n\cos(n\theta) + B_n\sin(n\theta)\right\}}$$

The coefficients are the Fourier coefficients of $f(\theta)$:

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta, \quad A_n a^{-n} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$B_n a^{-n} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta$$