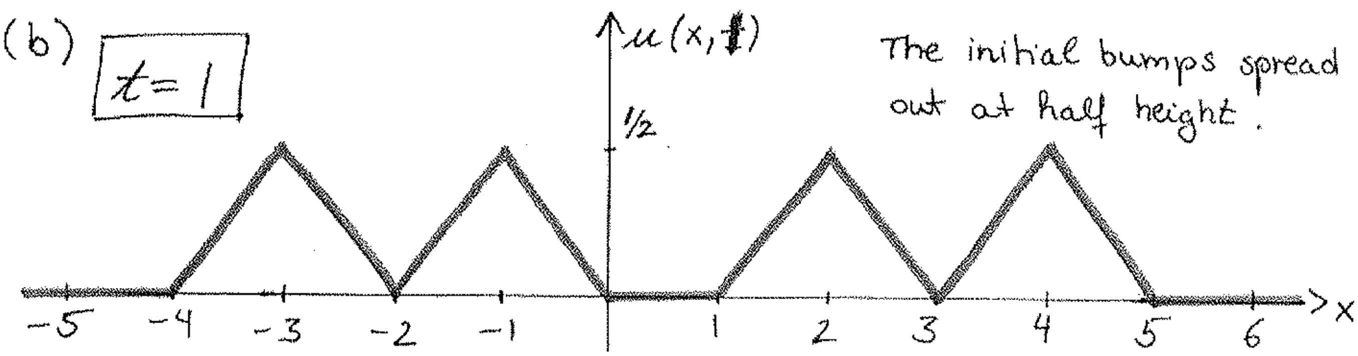
Problem 1:

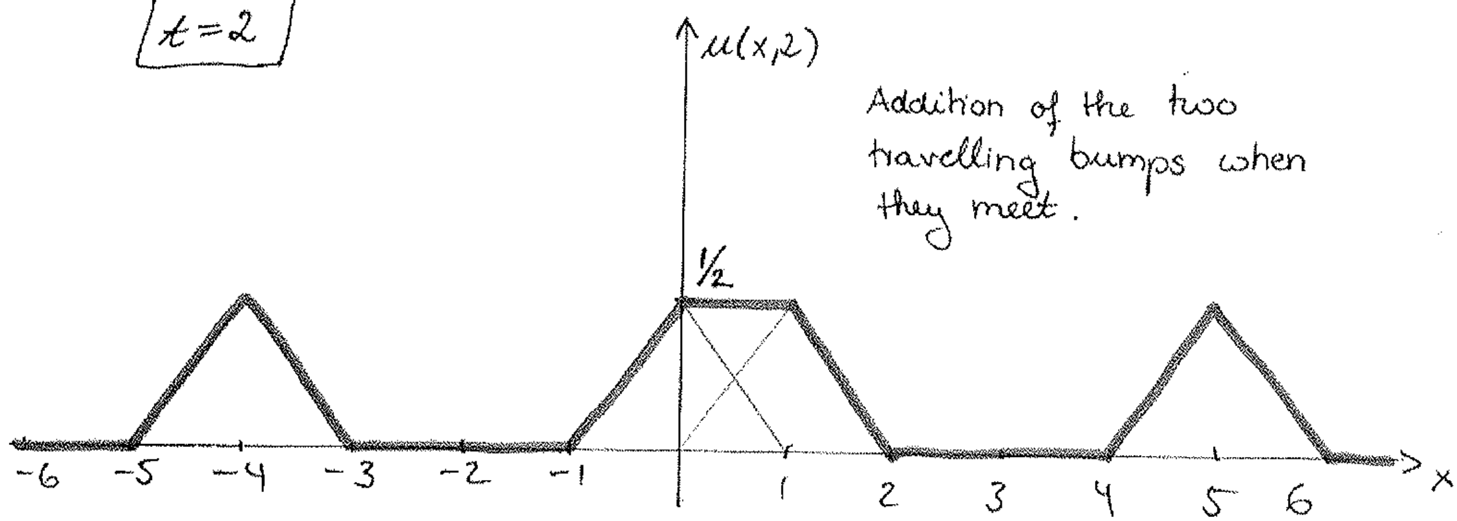
(a) $u(x,t) = \frac{1}{2}\left\{ f(x-t) + f(x+t) \right\} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$

(b) $\boxed{t=1}$



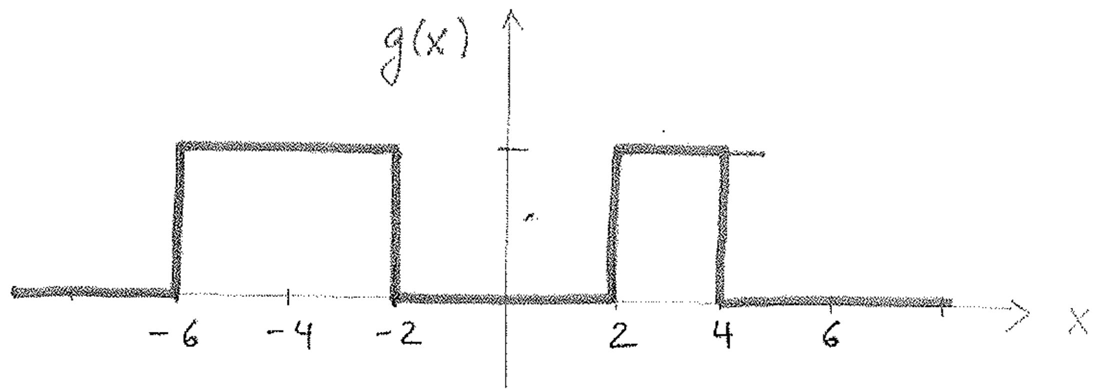
The initial bumps spread out at half height.

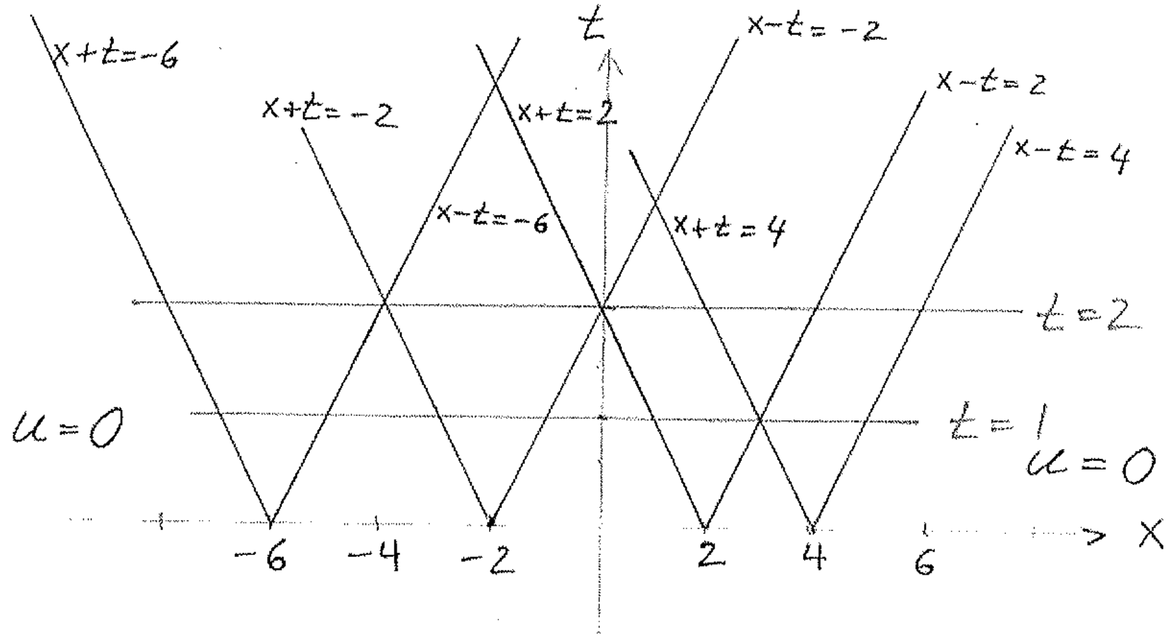
$\boxed{t=2}$



Addition of the two travelling bumps when they meet.

(c)





$\underline{x = -6-t \,..\, -2-t}$

$\underline{t=1:}$ $\quad x = -7 \,..\, -5$

$$u(x,1) = \frac{1}{2}\int_{x-1}^{-6} 0\, ds + \frac{1}{2}\int_{-6}^{x+1} 1\, ds = \frac{1}{2}x + \frac{7}{2}$$

$\underline{t=2:}$ $\quad x = -8 \,..\, -4$

$$u(x,2) = \frac{1}{2}\int_{x-2}^{-6} 0\, ds + \frac{1}{2}\int_{-6}^{x+2} 1\, ds = \frac{1}{2}x + 4$$

$x = -6+t \,..\, -2-t$

$t=1$: $x = -5 \,..\, -3$

$$u(x,1) = \frac{1}{2}\int_{x-1}^{x+1} 1\, ds = 1$$

$t=2$: $x = -4$

$$u(x,2) = \frac{1}{2}\int_{x-2}^{x+2} 1\, ds = 2$$

$x = -2-t \,..\, -2+t$

$t=1$: $x = -3 \,..\, -1$

$$u(x,1) = \frac{1}{2}\int_{x-1}^{-2} 1\, ds + \frac{1}{2}\int_{-2}^{x+1} 0\, ds = -\frac{1}{2}x - \frac{1}{2}$$

$t=2$: $x = -4 \,..\, 0$

$$u(x,2) = \frac{1}{2}\int_{x-2}^{-2} 1\, ds + \frac{1}{2}\int_{-2}^{x+2} 0\, ds = -\frac{1}{2}x$$

$x = -2+t \,..\, 2-t$

$t=1$: $x = -1 \,..\, 1$

$$u(x,1) = 0$$

$t=2$: $x = 0$

$$u(x,2) = 0$$

$x = 2-t \,..\, 2+t$

$t=1$: $x = 1 \,..\, 3$

$$u(x,1) = \frac{1}{2}\int_{x-1}^{2} 0\,ds + \frac{1}{2}\int_{2}^{x+1} 1\,ds = \frac{1}{2}x - \frac{1}{2}$$

$t=2$: $x = 0 \,..\, 4$

$$\tilde{u}(x,2) = \frac{1}{2}\int_{x-2}^{2} 0\,ds + \frac{1}{2}\int_{2}^{x+2} 1\,ds = \frac{1}{2}x$$

$x = 4-t \,...\, 4+t$

$t=1$: $x = 3 \,..\, 5$

$$u(x,1) = \frac{1}{2}\int_{x-1}^{4} 1\,ds + \frac{1}{2}\int_{4}^{x+1} 0\,ds = -\frac{x}{2} + \frac{5}{2}$$

$t=2$: $x = 2 \,...\, 6$

$$\hat{u}(x,2) = \frac{1}{2}\int_{x-2}^{4} 1\,ds + \frac{1}{2}\int_{4}^{x+2} 0\,ds = -\frac{x}{2} + 3$$

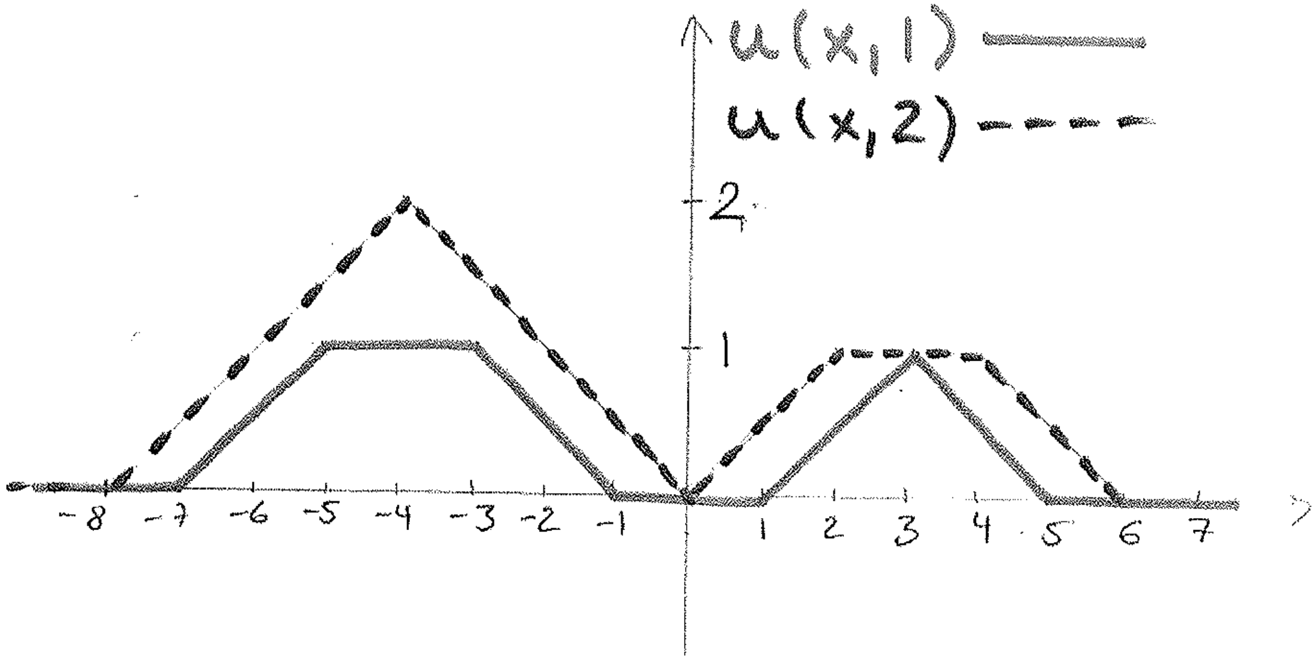
Note: For $t=2$ the marked regions overlap.

Therefore we get

$$u(x,2) = \tilde{u}(x,t) = \frac{1}{2}x \quad \text{for } x = 0 \,..\, 2$$

$$u(x,2) = \tilde{u}(x,t) + \hat{u}(x,t) = 2 \quad \text{for } x = 2 \,..\, 4$$

$$u(x,2) = \hat{u}(x,t) = -\frac{x}{2} + 3 \quad \text{for } x = 4 \,..\, 6$$



u(x,1)
u(x,2)
2
1
-8
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
7

Problem 2: $u_{tt} = c^2 u_{xx}$, $0<x<1$, $t>0$

$$u(0,t) = 0 = u(1,t)$$

$$u(x,0) = 2\sin(2\pi x) + 3\sin(3\pi x)$$

$$u_t(x,0) = \sin(\pi x) \quad , \quad 0<x<1$$

Assume $u(x,t) = X(x)\,T(t)$

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} =: -\lambda^2$$

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(1) \end{cases}$$

$\underline{\lambda = 0}$: trivial solution

$\underline{\lambda > 0}$: $X(x) = A\cos\lambda x + B\sin\lambda x$

$$X(0) = A \overset{!}{=} 0 \Rightarrow A = 0$$

$$X(1) = B\sin\lambda \overset{!}{=} 0$$

$$\Rightarrow \lambda_n = n\pi \quad \text{for } n = 1, 2, \dots$$

$$X_n = \sin(n\pi x) \quad \text{for } n = 1, 2, \dots$$

$$\ddot{T} + c^2\lambda^2 T = 0$$

$$T_n(t) = C_n\cos(n\pi c t) + D_n\sin(n\pi c t) \quad , \quad n = 1, 2, \dots$$

$$\Rightarrow T_n(0) = C_n$$

$$u_n(x,t) = T_n(t)\, X_n(x)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left\{ -n\pi c\, C_n \sin(n\pi c t) + n\pi c\, D_n \cos(n\pi c t) \right\} \sin(n\pi x)$$

Initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) \overset{!}{=} 2\sin(2\pi x) + 3\sin(3\pi x)$$

$$\Rightarrow \quad C_n = \begin{cases} 2 & \text{if } n=2 \\ 3 & \text{if } n=3 \\ 0 & \text{otherwise} \end{cases}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} n\pi c\, D_n \sin(n\pi x) \overset{!}{=} \sin \pi x$$

$$\Rightarrow \quad D_n = \begin{cases} (n\pi c)^{-1} & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$$

$$u(x,t) = \frac{1}{\pi c} \sin(\pi c t) \sin(\pi x) + 2\cos(2\pi c t)\sin(2\pi x)$$

$$+ 3\cos(3\pi c t)\sin(3\pi x)$$

Problem 3

Separation yields

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\mu^2, \quad \mu \geq 0$$

The eigenvalue problem in space is

$$X'' + \mu^2 X = 0, \quad X'(0) = X'(L) = 0$$

The eigenvalues / eigenfunctions are:

- $\mu_0 = 0, \quad X_0 = \text{const}$
- $\mu_n^2 = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$

  $X_n = \cos\left(\frac{n\pi}{L} x\right)$

In time, we obtain the problem

$$T'' + c^2 \mu^2 T = 0$$

Hence, we have

- $\mu_0 = 0 \Rightarrow T'' = 0 \Rightarrow T(t) = \text{linear} = \frac{B_0}{2} + \frac{A_0}{2} t$
- $\mu_n^2 = \left(\frac{n\pi}{L}\right)^2 \Rightarrow T(t) = A_n \sin\left(\frac{n\pi}{L} c t\right) + B_n \cos\left(\frac{n\pi}{L} c t\right)$

Hence, the general solution is of the form

$$\underline{\underline{u(x,t) = \frac{B_0}{2} + \frac{A_0}{2} t}}$$

$$\underline{\underline{+ \sum_{n=1}^{\infty} \left(A_n \sin\left(\frac{n\pi}{L} c t\right) + B_n \cos\left(\frac{n\pi}{L} c t\right) \right) \cos\left(\frac{n\pi}{L} x\right)}}$$

$$u(x,0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{L} x\right) \overset{!}{=} f(x)$$

$\Rightarrow$ $B_0, B_1, \ldots, B_n, \ldots$ are the Fourier cosine coefficients of $f(x)$

$$u_t(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (-A_n) \frac{n\pi c}{L} \cos\left(\frac{n\pi}{L} x\right) \overset{!}{=} g(x)$$

$\Rightarrow$ $A_0, A_1, \ldots, A_n, \ldots$ can be determined by the Fourier sine coefficients of $g(x)$.