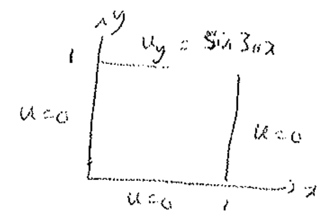
# Assignment 9

1. $u_{xx} + u_{yy} + u = 0$

$u(0,y) = 0 \qquad u(1,y) = 0$

$u(x,0) = 0 \qquad u_y(x,1) = \sin 3\pi x.$




$u = \cancel{XY}$ $\Rightarrow$ $X''Y + XY'' + XY = 0$
$X(x)Y(y)$

$$\Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} - 1 = -\lambda^2$$

$X'' + \lambda^2 X = 0 \qquad X(0) = X(1) = 0 \qquad \Rightarrow \quad \boxed{X = B\sin\lambda x}$ and $\boxed{\lambda = n\pi.}$

$Y'' = (\lambda^2 - 1)Y \qquad Y(0) = 0 \qquad \Rightarrow \quad Y = C\cosh\sqrt{\lambda^2-1}\,y + D\sinh\sqrt{\lambda^2-1}\,y \qquad (\lambda \geq \pi).$

& $Y(0) = 0 \Rightarrow C = 0$ so $\boxed{Y = D\sinh\sqrt{\lambda^2-1}\,y}$

$\therefore$ Possible solutions are $u = XY = A_n \sin(n\pi x)\sinh\left((n^2\pi^2-1)^{1/2}y\right)$

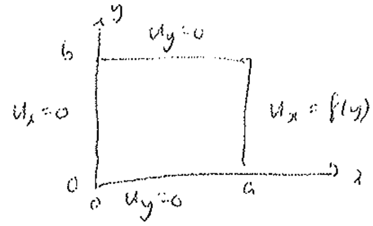
From the final condition $u_y(x,1) = \sin 3\pi x$ it is clear the $n=3$ solution is required, and

$$u_y = A_n\sqrt{n^2\pi^2-1}\,\sin(n\pi x)\cosh\left(\sqrt{n^2\pi^2-1}\,y\right)$$

$$\Rightarrow \text{ need } \quad A_3\sqrt{9\pi^2-1}\,\cosh\sqrt{9\pi^2-1} = 1 \quad \Rightarrow A_3 = \frac{1}{\sqrt{9\pi^2-1}\,\cosh\sqrt{9\pi^2-1}}$$

$$\Rightarrow \quad u = \frac{\sinh\left(\sqrt{9\pi^2-1}\,y\right)}{\sqrt{9\pi^2-1}\,\cosh\sqrt{9\pi^2-1}}\sin 3\pi x$$

2. $u_{xx} + u_{yy} = 0$




a) $u = X(x)Y(y) \Rightarrow X''Y + Y''X = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2.$

Boundary conditions require $Y'(0) = Y'(b) = 0 \quad X'(0) = 0.$

$Y'' + \lambda^2 Y = 0 \Rightarrow Y = A\cos\lambda y + B\sin\lambda y \quad Y'(0)=0 \Rightarrow B=0 \quad Y'(b)=0 \Rightarrow \boxed{\lambda = \frac{n\pi}{b}}$

$(n = 0, 1, 2, \ldots) \quad \boxed{Y = A\cos\frac{n\pi y}{b}}$

$X'' - \lambda^2 X = 0 \Rightarrow X = C\cosh\lambda x + D\sinh\lambda x \quad (\lambda > 0)$

$X'(0) = 0 \Rightarrow D = 0 \Rightarrow \boxed{X = C\sinh\frac{n\pi x}{b}} \quad (n > 0)$

$\lambda = 0$ case $\Rightarrow X = \hat{C}x + \hat{D}$

$X'(0) = 0 \Rightarrow \hat{C} = 0 \Rightarrow \boxed{X = \hat{D}} \quad (n = 0).$

Combining all possible values of $n$, general solution is

$$\boxed{u(x,y) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right)}$$

(The naming of the new constants $A_n$ is arbitrary).

b). If $u(x,y)$ is a solution consider $w(x,y) = u + c$, where $c$ is a constant.

$\Rightarrow w_{xx} + w_{yy} = u_{xx} + u_{yy} = 0$ ✓

$w_x = u_x$ so $w_x(0,y) = 0$, $w_x(a,y) = f(y)$ ✓

$w_y = u_y$ so $w_y(x,0) = 0$, $w_y(x,b) = 0$ ✓

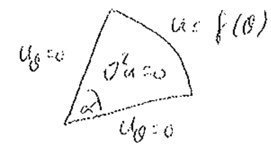
$\therefore$ $w$ satisfies the equation and boundary conditions and is therefore also a solution.

c) $f(y) = u_x(a,y) = \sum_{n=1}^{\infty} \underbrace{A_n \frac{n\pi}{b} \sinh\left(\frac{n\pi a}{b}\right)}_{\text{These must match Fourier cosine series } f(y) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi y}{b}\right)} \cos\left(\frac{n\pi y}{b}\right).$

Comparing the coefficients, this is only possible if $a_0 = 0 \Rightarrow \int_0^b f(y)\,dy = 0$

in which case $A_n = \dfrac{a_n}{\frac{n\pi}{b}\sinh\left(\frac{n\pi a}{b}\right)} = \boxed{\dfrac{2}{n\pi\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y)\cos\left(\frac{n\pi y}{b}\right)dy}.$

3. $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ $\qquad u_\theta(r,0) = 0 \quad u_\theta(r,\alpha) = 0$




$u = R(r)\,\Theta(\theta)$

$\Rightarrow \Theta R'' + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$

$\Rightarrow \dfrac{r^2R'' + rR'}{R} = -\dfrac{\Theta''}{\Theta} = \lambda^2$

$\Theta'' = -\lambda^2\Theta \qquad \Theta'(0) = \Theta'(\alpha) = 0.$

$\Rightarrow \Theta = A\cos\lambda\theta + B\sin\lambda\theta$

$\Theta' = -A\lambda\sin\lambda\theta + B\lambda\cos\lambda\theta$

$\Theta'(0) = 0 \Rightarrow B = 0, \qquad \Theta'(\alpha) = 0 \Rightarrow \lambda\alpha = n\pi$

$\boxed{\Theta = A\cos\frac{n\pi\theta}{\alpha}} \qquad \boxed{\lambda = \frac{n\pi}{\alpha}}$

$r^2R'' + rR' - \lambda^2 R = 0$

$R \propto r^m \Rightarrow \quad m^2 - m + m - \lambda^2 = 0$

$\Rightarrow \quad m = \pm\lambda$

$\Rightarrow \quad (\lambda > 0) \quad R = Cr^\lambda + Dr^{-\lambda}$

$D = 0$ so that $R$ is bounded at origin $\Rightarrow \boxed{R = Cr^\lambda} \qquad \lambda > 0$

$\lambda = 0 \Rightarrow R = \hat{C}\ln r + \hat{D}$ $\qquad$ [since $r^2R'' + rR' = 0 \Rightarrow r(rR')' = 0 \Rightarrow R' = \frac{\hat{C}}{r} \Rightarrow \hat{R} = \hat{C}\ln r + \hat{D}$]

$\hat{C} = 0$ so that $R$ is bounded at origin $\Rightarrow \boxed{R = \hat{D}} \qquad \lambda = 0.$

Summing all possible values of $n$, general solution is

$$\boxed{u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^{n\pi/\alpha}\cos\frac{n\pi\theta}{\alpha}}$$

$f(\theta) = u(a,\theta) = A_0 + \sum_{n=1}^{\infty} A_n a^{n\pi/\alpha}\cos\frac{n\pi\theta}{\alpha}$

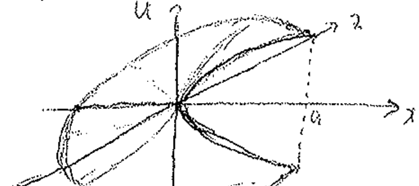
comparing with Fourier cosine coefficients for $f(\theta) \Rightarrow$ $\boxed{A_0 = \frac{1}{\alpha}\int_0^\alpha f(\theta)\,d\theta}$

$$\boxed{A_n = \frac{2}{\alpha\, a^{n\pi/\alpha}}\int_0^\alpha f(\theta)\cos\frac{n\pi\theta}{\alpha}\,d\theta \quad (n > 0)}$$

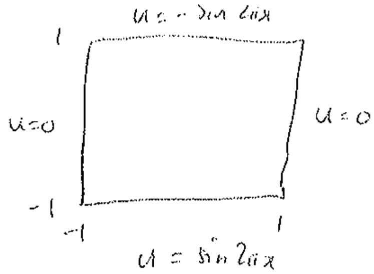
If $f(\theta) = \cos\frac{\theta}{2}$ and $\alpha = 2\pi$ then just need $n = 1$ component of solution and $A_1 = a^{-1/2}$.

(just choose $A_1$ so that $u(a,\theta) = \cos\frac{\theta}{2}$).

$$\boxed{u(r,\theta) = \left(\frac{r}{a}\right)^{1/2}\cos\frac{\theta}{2}}$$



4. $u_{xx} + u_{yy} = 0$




$u = X(x)\,Y(y)$

$\Rightarrow X''Y + Y''X = 0$

$\Rightarrow \dfrac{X''}{X} = -\dfrac{Y''}{Y} = -\lambda^2.$

$X'' + \lambda^2 X = 0 \qquad X(-1) = X(1) = 0.$

$\Rightarrow X = A\cos\lambda x + B\sin\lambda x$

$$\left.\begin{aligned} X(-1) &= A\cos\lambda - B\sin\lambda = 0 \\ X(1) &= A\cos\lambda + B\sin\lambda = 0 \end{aligned}\right\}$$ add $\Rightarrow A\cos\lambda = 0$

$\Rightarrow A = 0$ or $\lambda = (n+\tfrac{1}{2})\pi$

subtract $\Rightarrow B\sin\lambda = 0$

$\Rightarrow B = 0$ or $\lambda = n\pi.$

The other boundary conditions suggest we need $\lambda = 2\pi$ soln.

$Y'' - \lambda^2 Y = 0 \qquad Y(1) = -Y(-1)$ (from looking at form of boundary conditions).

$\Rightarrow Y = C\cosh\lambda y + D\sinh\lambda y$

$Y(1) = -Y(-1) \Rightarrow \quad C\cosh\lambda y + D\sinh\lambda y = -C\cosh\lambda y + D\sinh\lambda y$

$\Rightarrow C = 0 \qquad \boxed{Y = D\sinh\lambda y}$

Taking $\lambda = 2\pi$, (so must have $A = 0$ and $X = B\sin 2\pi x$)

$\Rightarrow \quad u = B\sin(2\pi x)\sinh(2\pi y)$

& to satisfy boundary conditions, $B = \dfrac{-1}{\sinh 2\pi} \Rightarrow \boxed{u = -\dfrac{\sinh(2\pi y)\sin 2\pi x}{\sinh 2\pi}}$

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -2E-16 | -0.5878 | -0.9511 | -0.9511 | -0.5878 | -8E-16 | 0.5878 | 0.9511 | 0.9511 | 0.5878 | 9E-16 | -0.5878 | -0.9511 | -0.9511 | -0.5878 | -1E-15 | 0.5878 | 0.9511 | 0.9511 | 0.5878 | 2E-15 |
| 0.9 | 0 | -0.3199 | -0.5176 | -0.5177 | -0.3201 | -0.0003 | 0.3195 | 0.5171 | 0.5171 | 0.3194 | -0.0004 | -0.3202 | -0.5179 | -0.5179 | -0.3202 | -0.0003 | 0.3195 | 0.5173 | 0.5173 | 0.3197 | 0 |
| 0.8 | 0 | -0.1742 | -0.2818 | -0.2819 | -0.1745 | -0.0006 | 0.1734 | 0.2809 | 0.2808 | 0.1733 | -0.0008 | -0.1748 | -0.2823 | -0.2823 | -0.1747 | -0.0006 | 0.1735 | 0.2811 | 0.2813 | 0.1739 | 0 |
| 0.7 | 0 | -0.0949 | -0.1536 | -0.1537 | -0.0954 | -0.0008 | 0.0938 | 0.1522 | 0.1521 | 0.0936 | -0.0011 | -0.0958 | -0.1543 | -0.1543 | -0.0957 | -0.0009 | 0.0939 | 0.1526 | 0.1528 | 0.0945 | 0 |
| 0.6 | 0 | -0.0517 | -0.0838 | -0.084 | -0.0524 | -0.001 | 0.0504 | 0.0821 | 0.082 | 0.0501 | -0.0014 | -0.0529 | -0.0847 | -0.0847 | -0.0527 | -0.0011 | 0.0505 | 0.0826 | 0.0825 | 0.0512 | 0 |
| 0.5 | 0 | -0.0283 | -0.0458 | -0.0461 | -0.029 | -0.0012 | 0.0267 | 0.0438 | 0.0437 | 0.0264 | -0.0016 | -0.0286 | -0.0460 | -0.0468 | -0.0294 | -0.0013 | 0.0269 | 0.0444 | 0.0447 | 0.0277 | 0 |
| 0.4 | 0 | -0.0155 | -0.0251 | -0.0254 | -0.0162 | -0.0013 | 0.0137 | 0.0229 | 0.0228 | 0.0134 | -0.0018 | -0.0169 | -0.0263 | -0.0262 | -0.0167 | -0.0014 | 0.0139 | 0.0236 | 0.0239 | 0.0148 | 0 |
| 0.3 | 0 | -0.0084 | -0.0137 | -0.014 | -0.0092 | -0.0014 | 0.0066 | 0.0114 | 0.0113 | 0.0062 | -0.0018 | -0.01 | -0.015 | -0.0149 | -0.0098 | -0.0015 | 0.0058 | 0.0121 | 0.0124 | 0.0076 | 0 |
| 0.2 | 0 | -0.0045 | -0.0073 | -0.0076 | -0.0053 | -0.0014 | 0.0025 | 0.005 | 0.0048 | 0.0022 | -0.0019 | -0.0061 | -0.0086 | -0.0085 | -0.0058 | -0.0015 | 0.0020 | 0.0057 | 0.006 | 0.0038 | 0 |
| 0.1 | 0 | -0.0021 | -0.0035 | -0.0037 | -0.0029 | -0.0014 | 0.0002 | 0.0011 | 0.001 | -0.0002 | -0.0019 | -0.0037 | -0.0047 | -0.0046 | -0.0034 | -0.0015 | 0.0005 | 0.0018 | 0.0021 | 0.0014 | 0 |
| -1E-16 | 0 | -0.0003 | -0.0006 | -0.0009 | -0.0011 | -0.0014 | -0.0015 | -0.0017 | -0.0018 | -0.0018 | -0.0019 | -0.0019 | -0.0018 | -0.0018 | -0.0016 | -0.0014 | -0.0012 | -0.0009 | -0.0006 | -0.0003 | 0 |
| -0.1 | 0 | 0.0014 | 0.0022 | 0.0019 | 0.0006 | -0.0013 | -0.0032 | -0.0044 | -0.0045 | -0.0035 | -0.0018 | -6E-05 | 0.001 | 0.0011 | 0.0002 | -0.0014 | -0.0029 | -0.0037 | -0.0034 | -0.002 | 0 |
| -0.2 | 0 | 0.0038 | 0.0061 | 0.0059 | 0.0031 | -0.0012 | -0.0055 | -0.0061 | -0.0082 | -0.0056 | -0.0017 | 0.0025 | 0.005 | 0.0051 | 0.0027 | -0.0012 | -0.0052 | -0.0075 | -0.0072 | -0.0044 | 0 |
| -0.3 | 0 | 0.0078 | 0.0126 | 0.0124 | 0.0072 | -0.0011 | -0.0093 | -0.0144 | -0.0145 | -0.0096 | -0.0015 | 0.0066 | 0.0116 | 0.0117 | 0.0068 | -0.0011 | -0.009 | -0.0138 | -0.0136 | -0.0083 | 0 |
| -0.4 | 0 | 0.0149 | 0.0241 | 0.0239 | 0.0143 | -0.001 | -0.0162 | -0.0257 | -0.0257 | -0.0164 | -0.0013 | 0.0138 | 0.0232 | 0.0233 | 0.014 | -0.001 | -0.016 | -0.0251 | -0.0249 | -0.0154 | 0 |
| -0.5 | 0 | 0.0278 | 0.0449 | 0.0447 | 0.0273 | -0.0008 | -0.0289 | -0.0463 | -0.0463 | -0.0291 | -0.0011 | 0.0269 | 0.0442 | 0.0442 | 0.027 | -0.0008 | -0.0287 | -0.0458 | -0.0456 | -0.0282 | 0 |
| -0.6 | 0 | 0.0513 | 0.083 | 0.0829 | 0.0509 | -0.0007 | -0.0522 | -0.0841 | -0.0842 | -0.0524 | -0.0009 | 0.0506 | 0.0824 | 0.0825 | 0.0507 | -0.0007 | -0.052 | -0.0837 | -0.0836 | -0.0516 | 0 |
| -0.7 | 0 | 0.0946 | 0.153 | 0.1528 | 0.0943 | -0.0005 | -0.0952 | -0.1538 | -0.1538 | -0.0953 | -0.0007 | 0.094 | 0.1525 | 0.1526 | 0.0941 | -0.0005 | -0.0951 | -0.1535 | -0.1534 | -0.0948 | 0 |
| -0.8 | 0 | 0.1739 | 0.2814 | 0.2813 | 0.1737 | -0.0003 | -0.1744 | -0.262 | -0.262 | -0.1745 | -0.0005 | 0.1736 | 0.2811 | 0.2811 | 0.1736 | -0.0003 | -0.1743 | -0.2818 | -0.2817 | -0.1741 | 0 |
| -0.9 | 0 | 0.3198 | 0.5174 | 0.5174 | 0.3197 | -0.0002 | -0.32 | -0.5177 | -0.5177 | -0.32 | -0.0002 | 0.3196 | 0.5173 | 0.5173 | 0.3196 | -0.0002 | -0.32 | -0.5176 | -0.5175 | -0.3199 | 0 |
| -1 | 2E-16 | 0.5878 | 0.9511 | 0.9511 | 0.5878 | 8E-16 | -0.5878 | -0.9511 | -0.9511 | -0.5878 | -9E-16 | 0.5878 | 0.9511 | 0.9511 | 0.5878 | 1E-15 | -0.5878 | -0.9511 | -0.9511 | -0.5878 | -2E-15 |
| Y / X | -1 | -0.9 | -0.8 | -0.7 | -0.6 | -0.5 | -0.4 | -0.3 | -0.2 | -0.1 | -1E-16 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |

dx 0.1
dy 0.1