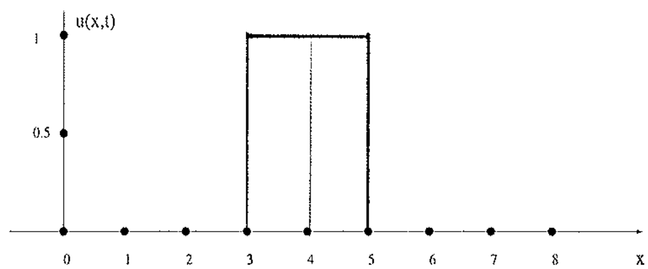
Problem 1 (10 points)

Consider the wave equation:

$$
\begin{aligned}
&u_{tt} = 4u_{xx}, \qquad 0 < x < 8,\ t > 0,\\
&u(0,t) = 0, \quad u(8,t) = 0,\\
&u(x,0) = f(x) = \begin{cases} 1 & 3 \le x \le 5,\\ 0 & \text{otherwise.}\end{cases}\\
&u_t(x,0) = 0.
\end{aligned}
$$

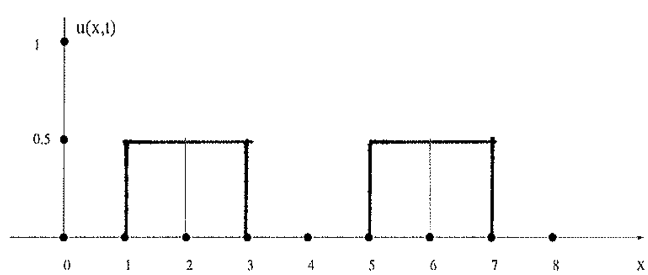
In the coordinate systems provided below, carefully sketch the solution $u(x,t)$ for $t = 0$, $t = 1$, $t = 2$, and $t = 3$.

a) $t = 0$ [2]



wave speed = 2.

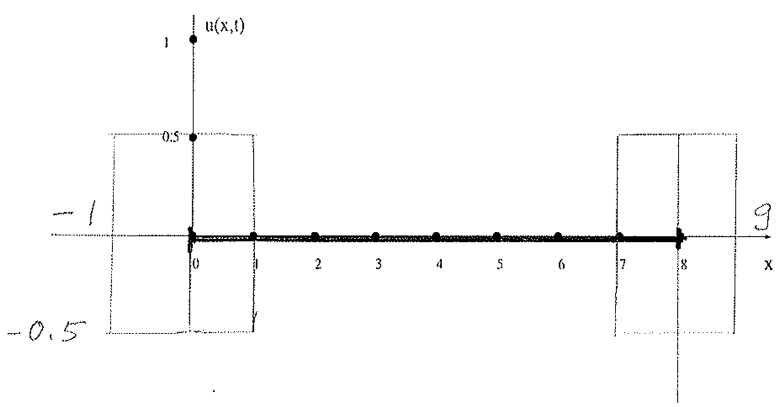
b) $t = 1$ [2]



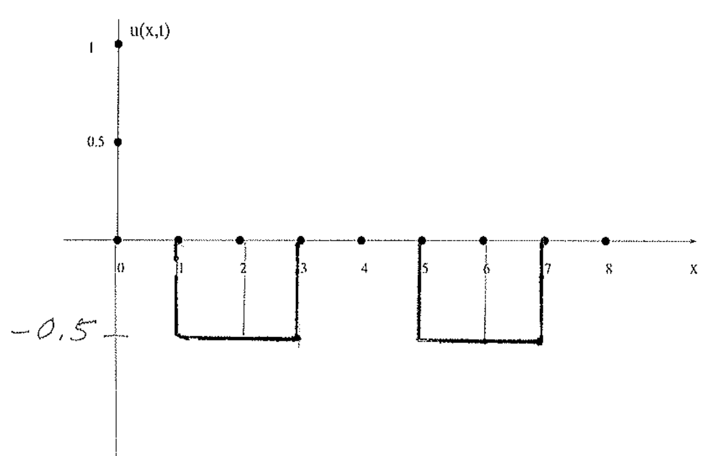
Problem 1 (continued)

$u(x,t) = 0$

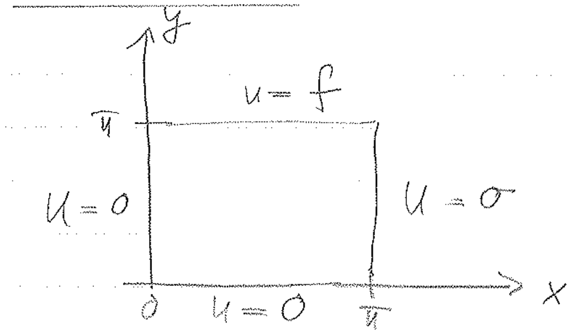
c) $t = 2$ [3]



d) $t = 3$ [3]



Problem 2:



$$u(x,y) = X(x)\,Y(y)$$

homogeneous in x

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\mu^2, \quad \mu \geq 0$$

$\boxed{\text{in } x:}$ $\quad X'' + \mu^2 X = 0$, $\quad X(0) = 0$, $\quad X(\pi) = 0$ $\quad \Big\}\quad \mu_n^2 = n^2, \quad X_n = \sin(nx)$, $\quad n = 1, 2, \ldots$

$\boxed{\text{in } y:}$ $\quad Y'' - \mu^2 Y = 0$, $\quad Y(0) = 0$, $\quad Y(\pi) = \text{undetermined}$

$\mu_n = n$:

$$Y_n(y) = A_n \sinh(ny) + B_n \cosh(ny)$$

$$Y_n(0) = 0 \Rightarrow \underline{B_n = 0}$$

$$\boxed{\Rightarrow u(x,y) = \sum_{n=1}^{\infty} A_n \sin(nx)\sinh(ny) \quad \text{where } A_n = \frac{1}{\sinh(n\pi)}\,\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx}$$

Problem 3:

a) Comparing to the standard form $x^2y'' + x(xp(x))y' + x^2qy = 0$ we have that

$xp(x) = 2 - x$

$x^2q(x) = 3/2\,x - 3/4$

both analytic for $x = 0$

b) $$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}$$

$$\Rightarrow x^2y'' + 2xy' - x^2y' + \frac{3}{2}xy - \frac{3}{4}y = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + 2\sum_{n=0}^{\infty} (n+r) a_n x^{n+r}$$

$$\underbrace{- \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} + \frac{3}{2}\sum_{n=0} a_n x^{n+r+1}}_{} - \frac{3}{4}\sum_{n=0} a_n x^{n+r} = 0$$

$$= - \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} + \frac{3}{2}\sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

$n=0$: $\left(r(r-1) + 2r - \frac{3}{4}\right) a_0 = 0$ indicial equation

$$\Rightarrow r^2 + r - \frac{3}{4} = 0$$

$$r_{1,2} = \frac{-1 \pm \sqrt{1 + 4\cdot 1\cdot 3/4}}{2} = \frac{-1 \pm 2}{2} \begin{cases} 1/2 \\ -\frac{3}{2} \end{cases}$$

singular exponent: $r_1 = 1/2$, $r_2 = -\frac{3}{2}$

for $n \geq 1$, we have the recurrence relation

$$\underbrace{\left((n+r)(n+r-1)+2(n+r)-\frac{3}{4}\right)}_{\left((n+r)(n+r+1)-3/4\right)} a_n + \underbrace{\left(-(n+r-1)+\frac{3}{2}\right)}_{-(n+r)+\frac{5}{2}} a_{n-1} = 0$$

c) $\underline{r = 1/2}$: $(n+r)(n+r+1) - 3/4 = \left(n+1/2\right)\left(n+3/2\right) - 3/4$

$$= n^2 + \frac{n}{2} + \frac{3n}{2} + \frac{3}{4} - 3/4 = n^2 + 2n$$

$$= n(n+2)$$

$$-(n+r) + \frac{5}{2} = -n + 2 = -(n-2)$$

$$\Rightarrow \boxed{a_n = \frac{(n-2)\,a_{n-1}}{n(n+2)}}$$

$$a_0 = 1 \Rightarrow a_1 = \frac{-a_1}{1 \cdot 3} = -\frac{1}{3}$$

$\Rightarrow a_2 = 0 \Rightarrow a_3 = 0 \ldots$ and so on.

$$\underline{\underline{y(x) = x^{1/2}\left(1 - \frac{1}{3}x\right)}}$$

Problem 4:

a)
$$\underline{\underline{a_0}} = \frac{2}{\pi}\int_0^{\pi} \frac{x}{\pi}\,dx$$

$$= \frac{x^2}{\pi^2}\Big|_0^{\pi} = \underline{\underline{1}}$$

$$\underline{\underline{a_n}} = \frac{2}{\pi}\int_0^{\pi} \overset{v}{\frac{x}{\pi}}\ \overset{u'}{\cos(nx)}\,dx$$

$$= \frac{2}{\pi^2}\left(\cancel{x\,\frac{\sin(nx)}{n}\Big|_0^{\pi}}^{\,=0} - \int_0^{\pi} \frac{\sin(nx)}{n}\,dx \right)$$

$$= \frac{2}{\pi^2}\,\frac{\cos(nx)}{n^2}\Big|_0^{\pi} = \underline{\underline{\frac{2}{\pi^2 n^2}\left(\cos(n\pi) - 1\right)}}$$

$$= \underline{\underline{\frac{2}{\pi^2 n^2}\left((-1)^n - 1\right)}}$$

$$= \begin{cases} -\dfrac{4}{\pi^2 n^2} & n = 1, 3, 5, 7, \ldots \\ 0 & n = 2, 4, 6, \ldots \end{cases}$$

$$\Rightarrow \underline{\underline{\frac{x}{\pi} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2}\left((-1)^n - 1\right)\cos(nx)}}$$

notice the sign

b) $u(r,\theta) = R(r)\Theta(\theta)$

homogeneous in θ ✓

$$\Rightarrow \frac{r^2 R'' + r R'}{R} = \boxed{-} \frac{\Theta''}{\Theta} = \boxed{+} \mu^2, \quad \mu \geq 0$$

$\boxed{\text{in } \theta:}$

$$\left.\begin{array}{l} \Theta'' + \mu^2 \Theta = 0 \\ \Theta'(0) = 0 \\ \Theta'(\pi) = 0 \end{array}\right\} \begin{array}{l} \mu_0^2 = 0, \ \Theta_0 = \text{const} \\ \mu_n^2 = n^2, \ \Theta_n = \cos(n\theta) \\ \qquad\qquad n = 1, 2, \dots \end{array}$$

$\boxed{\text{in } r:}$ $\quad r^2 R'' + r R' \boxed{-} \mu_n^2 R = 0$

$\underline{\mu_0 = 0:}$ $\quad R(r) = C_1 \log(r) + C_2$

boundedness as $r \to \infty$: $\quad R(r) = C_2$

$\underline{\mu_n = n:}$ $\quad R(r) = C_1 r^{+n} + C_2 r^{-n}$

boundedness as $r \to \infty$: $\quad R(r) = C_2 r^{-n}$

general form of the solution:

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n r^{-n} \cos(n\theta)$$

boundary condition:

$$u(1,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\theta) = \frac{\theta}{4}$$

From part a): $\quad A_0 = 1$

$$A_n = \frac{2}{\pi^2 n^2}\left((-1)^n - 1\right)$$

Problem 5:

$$u(x,t) = X(x)\,T(t)$$

$$\Rightarrow \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -\mu^2, \quad \mu \geq 0$$

$\boxed{\text{in } X:}$

$$\left.\begin{array}{l} X'' + \mu^2 X = 0 \\ X(-1) = X(1) \\ X'(-1) = X'(1) \end{array}\right\} \quad \begin{array}{l} \mu_0 = 0, \quad X_0 = \text{const} \\ \mu_n^2 = n^2\pi^2 \\ X_n = A_n \sin(n\pi x) + B_n \cos(n\pi x) \end{array}$$

$\boxed{\text{in } T:}$

$$T'' + c^2\mu^2 T = 0$$

$\underline{\mu_0 = 0}$: $T(t) = C_1 t + C_2$

$\underline{\mu_n = n}$: $T(t) = C_1 \sin(cn\pi t) + C_2 \cos(cn\pi t)$

General form of the solution:

$$u(x,t) = A_0 t + B_0 + \sum_{n=1}^{\infty} \left(A_n \sin(cn\pi t) + B_n \cos(cn\pi t)\right)\sin(n\pi x)$$
$$+ \sum_{n=1}^{\infty} \left(C_n \sin(cn\pi t) + D_n \cos(cn\pi t)\right)\cos(n\pi x)$$

$$u_t(x,t) = A_0 + \sum_{n=1}^{\infty}\left(A_n cn\pi \cos(cn\pi t) - B_n cn\pi \sin(cn\pi t)\right)\sin(n\pi x)$$
$$+ \sum_{n=1}^{\infty}\left(C_n cn\pi \cos(cn\pi t) - D_n cn\pi \cos(cn\pi t)\right)\cos(n\pi x)$$

Boundary conditions:

$$u(x,0) = B_0 + \sum_{n=1}^{\infty} B_n \sin(n\pi x) + \sum_{n=1}^{\infty} D_n \cos(n\pi x) \overset{!}{=} 1 + \cos(\pi x)$$

$$\Rightarrow \underline{B_0 = 1},\ B_1 = B_2 = \ldots = B_n = \ldots = 0,\ \underline{D_1 = 1},\ D_2 = 0 = \ldots D_n = 0$$

$$u_t(x,0) = A_0 + \sum_{n=1}^{\infty} A_n c n\pi \sin(n\pi x)$$
$$+ \sum_{n=1}^{\infty} C_n c n\pi \cos(n\pi x) \overset{!}{=} \sin(2\pi x)$$

$$\Rightarrow \underline{A_n = \begin{cases} 0 & n \neq 2 \\ \frac{1}{c2\pi} & n = 2 \end{cases}} \qquad \underline{C_n = 0}$$

In total:

$$\underline{u(x,t) = 1 + \cos(c\pi t)\cos(\pi x) + \frac{1}{2\pi c}\sin(2\pi c t)\sin(2\pi x)}$$

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