Problem 1: $L=1$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos n\pi x+b_n\sin n\pi x\right\}$$

$$a_0=\int_{-1}^{1}f(x)\,dx=\int_0^1(1-x)\,dx=x-\frac{x^2}{2}\Big|_0^1=\frac{1}{2}$$

$$a_n=\int_{-1}^{1}f(x)\cos(n\pi x)\,dx$$

$$=\int_0^1(1-x)\cos(n\pi x)\,dx=\frac{1}{n\pi}\sin n\pi x\Big|_0^1-\int_0^1 x\cos n\pi x\,dx$$

$$=\frac{1}{n\pi}\sin n\pi-\left[x\,\frac{1}{n\pi}\sin n\pi x\right]_0^1+\int_0^1\frac{1}{n\pi}\sin(n\pi x)\,dx$$

$$=\frac{-1}{(n\pi)^2}\cos n\pi x\Big|_0^1=\frac{1}{(n\pi)^2}\left(1-\cos n\pi\right)$$

$$=\frac{1}{n^2\pi^2}\left(1-(-1)^n\right)=\begin{cases}0 & \text{if } n \text{ even}\\ \frac{2}{n^2\pi^2} & \text{if } n \text{ odd}\end{cases}$$

$$b_n=\int_{-1}^{1}f(x)\sin(n\pi x)\,dx$$

$$=\int_0^1(1-x)\sin(n\pi x)\,dx=\frac{-1}{n\pi}\cos n\pi x\Big|_0^1-\int_0^1 x\sin(n\pi x)\,dx$$

$$=\frac{-1}{n\pi}\cos(n\pi)+\frac{1}{n\pi}-\left[x\left(-\frac{1}{n\pi}\cos n\pi x\right)\right]_0^1+\int_0^1\frac{1}{n\pi}\cos(n\pi x)\,dx$$

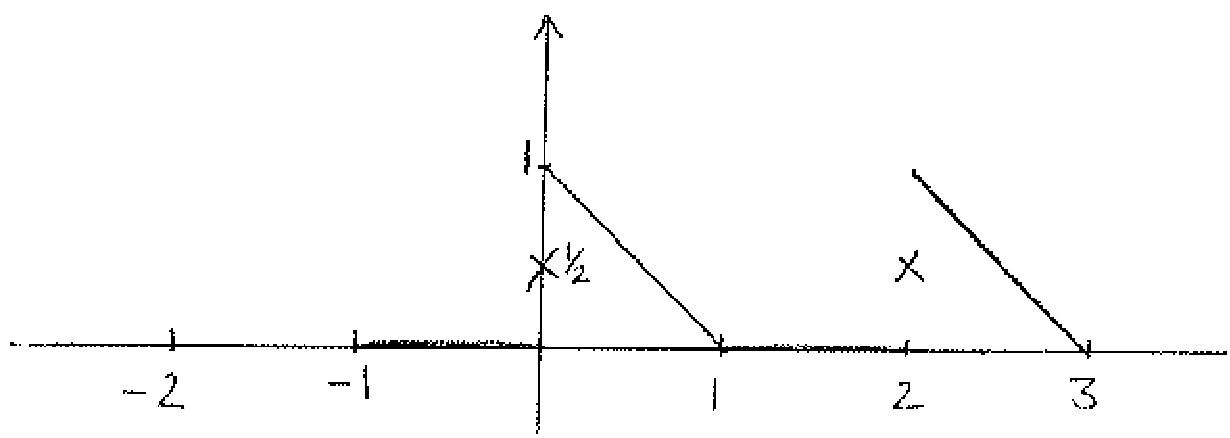
$$=\frac{1}{n\pi}+\left(\frac{1}{n\pi}\right)^2\sin(n\pi x)\Big|_0^1=\frac{1}{n\pi}$$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{n\pi} \left\{ \frac{1}{n\pi} \left(1-(-1)^n\right) \cos(n\pi x) + \sin(n\pi x) \right\}$$

OR

$$f(x) = \frac{1}{4} + \sum_{m=1}^{\infty} \frac{2}{(2m-1)^2 \pi^2} \cos\left((2m-1)\pi x\right)$$
$$+ \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin\left(n\pi x\right)$$

(b)



Problem 2:

(a) $P(x) = x^2$. Since $P(0) = 0$, x is a singular point.

Limits of $x p(x)$ and $x^2 q(x)$: $p(x) = \frac{x}{x^2}$, $q(x) = \frac{(x^2 - \frac{9}{4})}{x^2}$

$$\lim_{x \to 0} x p(x) = \lim_{x \to 0} 1 = 1 < \infty$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} \left(x^2 - \frac{9}{4}\right) = -\frac{9}{4} < \infty$$

$\Rightarrow$ $x = 0$ is regular singular point.

(b) $x > 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$$

$$\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n}$$

$$+ \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} \frac{9}{4} a_n x^{r+n} = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty} \left\{ (r+n)(r+n-1) + (r+n) - \frac{9}{4} \right\} a_n x^{r+n}$$

$$+ \sum_{n=2}^{\infty} a_{n-2} x^{r+n} = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty} \left[(r+n)^2 - \frac{9}{4} \right] a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} = 0$$

$$\left(r^2-\frac{9}{4}\right)a_0x^r+\left[(r+1)^2-\frac{9}{4}\right]a_1\,x^{r+1}$$

$$+\sum_{n=2}^{\infty}\left\{\left[(r+n)^2-\frac{9}{4}\right]a_n+a_{n-2}\right\}x^{r+n}=0$$

Indicial equation: $\left(r^2-\frac{9}{4}\right)=0$

$$\Rightarrow \boxed{r_1=\frac{3}{2}} \text{ and } \boxed{r_2=-\frac{3}{2}}$$

Since

$$\Big[\underbrace{(r+1)^2-\frac{9}{4}}_{\neq 0 \text{ for } r=\pm\frac{3}{2}}\Big]a_1 \overset{!}{=} 0 \quad\Rightarrow\quad \boxed{a_1=0}$$

Recurrence relation:

$$\boxed{a_n=\frac{-a_{n-2}}{(r+n)^2-\frac{9}{4}}}$$

(c) $r=\frac{3}{2}$: $a_n=\dfrac{-a_{n-2}}{\left(\frac{3}{2}+n\right)^2-\frac{9}{4}}=\dfrac{-a_{n-2}}{n(n+3)}$

$a_0=1$

$a_2=\dfrac{-1}{10}$

$a_4=\dfrac{-a_2}{28}=\dfrac{1}{280}$

$a_1=0 \Rightarrow a_3=a_5=\ldots=0$

$$y(x)=x^{3/2}\left(1-\frac{1}{10}x^2+\frac{1}{280}x^4-\ldots\right)$$

Problem 3:

(a) $0 = v_{xx} \Rightarrow v(x) = ax + b$

BC: $v(0) = b \overset{!}{=} 2 \Rightarrow b = 2$

$v(1) = a + 2 \overset{!}{=} 0 \Rightarrow a = -2$

$\Rightarrow v(x) = -2x + 2$

(b) $u(x,t) = v(x) + w(x,t)$

$w_t = 9 w_{xx}$

$u(0,t) = v(0) + w(0,t) = 2 + w(0,t) \overset{!}{=} 2 \Rightarrow w(0,t) = 0$

$u(1,t) = v(1) + w(2,t) = w(1,t) \overset{!}{=} 0 \Rightarrow w(1,t) = 0$

$u(x,0) = v(x) + w(x,0) \overset{!}{=} -2x \Rightarrow w(x,0) = -2x - (-2x + 2)$

$= -2$

$$\left\{ \begin{aligned} & w_t = 9 w_{xx} \\ & w(0,t) = 0 = w(1,t) \\ & w(x,0) = -2 \end{aligned} \right\}$$

$w(x,t) = X(x)\,T(t)$

Separation yields $\dfrac{\dot{T}(t)}{9T(t)} = \dfrac{X''}{X} =: -\lambda^2$

<u>time</u>: $\dot{T}(t) = -9\lambda^2 T(t)$

$\Rightarrow T(t) = c\, e^{-9\lambda^2 t}$

<u>space</u>: $\left\{ \begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 \;,\; X(1) = 0 \end{array} \right\}$

<u>$\lambda = 0$</u>: trivial solution

<u>$\lambda > 0$</u>: $X(x) = A\cos\lambda x + B\sin\lambda x$

$X(0) = A \overset{!}{=} 0 \Rightarrow A = 0$

$X(1) = B\sin\lambda \overset{!}{=} 0 \Rightarrow \lambda_n = n\pi$

eigenvalues: $\lambda_n^2 = (n\pi)^2 \quad , n = 1, 2, \ldots$

eigenfunctions: $X_n = \sin(n\pi x) \quad , n = 1, 2, \ldots$

$$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-9n^2\pi^2 t} \sin(n\pi x)$$

Initial condition

$$w(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \stackrel{!}{=} -2$$

$$\Rightarrow b_n = 2 \int_0^1 (-2) \sin n\pi x \, dx$$

$$= -4 \left(-\frac{1}{n\pi} \cos n\pi x\right)\Big|_0^1$$

$$= \frac{4}{n\pi} \left(\cos n\pi - 1\right)$$

$$= \frac{4}{n\pi} \left((-1)^n - 1\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{-8}{n\pi} & \text{if } n \text{ odd} \end{cases}$$

$$\Rightarrow u(x,t) = -2x + 2 + \sum_{n=1}^{\infty} \frac{(-8)}{(2n-1)\pi} e^{-(3(2n-1)\pi)^2 t} \sin((2n-1)\pi x)$$

Problem 4

a) $\underline{A_0} = \frac{6}{\pi} \int_0^{\pi/3} f(x)\,dx = \underline{0}$

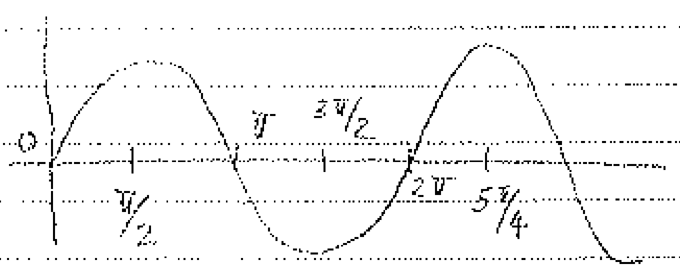
$$\underline{A_n} = \frac{6}{\pi} \int_0^{\pi/6} f(x) \cos(3nx)\,dx$$

$$= \frac{6}{\pi} \left(\int_0^{\pi/6} \cos(3nx)\,dx - \int_{\pi/6}^{\pi/3} \cos(3nx)\,dx \right)$$

$$= \frac{6}{\pi} \left(\frac{\sin(3nx)}{3n} \Bigg|_0^{\pi/6} - \frac{\sin(3nx)}{3n} \Bigg|_{\pi/6}^{\pi/3} \right)$$

$$= \frac{6}{\pi} \left(\frac{\sin\left(\frac{\pi n}{2}\right)}{3n} - \cancel{\frac{\sin(0)}{3n}} - \overbrace{\cancel{\frac{\sin(n\pi)}{3n}}}^{=0} + \frac{\sin\left(\frac{\pi n}{2}\right)}{3n} \right)$$

$$= \underline{\frac{4}{\pi n} \sin\left(\frac{n\pi}{2}\right)} = \frac{4}{\pi n} \begin{cases} 0 & n = 2, 4, 6, \ldots \\ 1 & n = 1, 5, 9, \ldots \\ -1 & n = 3, 7, 11, \ldots \end{cases}$$



$\Rightarrow$ We can write $\underline{A_{2k+1} = \frac{4(-1)^k}{\pi(2k+1)}}$ $\quad k = 0, 1, 2, \ldots$

or

$$A_{2k-1} = \frac{4(-1)^{k+1}}{\pi(2k-1)}, \quad k = 1, 2, 3, \ldots$$

b) $u(r,\theta) = R(r)\,\Theta(\theta)$

homogeneous in θ

Separation yields $\dfrac{r^2 R'' + r R'}{R} = -\dfrac{\Theta''}{\Theta} = \mu^2, \ \mu \geq 0$

$\boxed{\text{in } \theta:}$ $\Theta'' + \mu^2 \Theta = 0, \quad 0 < \theta < \frac{\pi}{3}$

$\Theta'(0) = 0$

$\Theta'(\frac{\pi}{3}) = 0$

$\Rightarrow$ eigenvalues: $\underline{\mu_n^2 = (3n)^2}$ $\quad n = 0, 1, 2, \ldots$

eigenfunctions: $\underline{\Theta_n = \cos(3n\theta)}$

$n = 0$ corresponds to the zero eigenvalue & constant eigenfunction

$\boxed{\text{in } r:}$ $r^2 R'' + r R' - \mu_n^2 R = 0$

boundedness

$\underline{n = 0}$: $\mu_0 = 0 \Rightarrow R(r) = A \ln(r) + B \Rightarrow \underline{R = \text{const} = \frac{A_0}{2}}$

boundedness

$\underline{n \geq 1}$: $\mu_n = 3n\pi \Rightarrow R(r) = A r^{\mu_n} + B r^{-\mu_n} \Rightarrow \underline{R = A r^{\mu_n}}$

The general solution is

$$\boxed{u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n r^{3n} \cos(3n\theta)}$$

Then: $u_r(1,\theta) = \sum_{n=1}^{\infty} A_n 3n \cos(3n\theta) \overset{!}{=} f(\theta)$

Since $\int_0^{\pi/3} f(\theta)\,d\theta = 0$, the left-hand side is consistent with a Fourier cosine series. We obtain from a)

$\underline{A_n = \frac{4}{3n^2\pi} \sin\left(\frac{n\pi}{2}\right)}$ or $\underline{A_{2k+1} = \frac{4(-1)^k}{3(2k+1)^2\pi}}$

$\boxed{\text{Problem 5:}}$ $u(x,y) = X(x)\,Y(y)$. Separation yields

$$\frac{X''}{X} = -\frac{Y''}{Y} = \mu^2 \quad \text{homogeneous}$$

$\boxed{\text{in } y:}$

$$\left.\begin{aligned} Y'' + \mu^2 Y &= 0 \\ Y(0) &= 0 \\ Y(\pi) &= 0 \end{aligned}\right\} \quad \begin{aligned} \mu_n^2 &= n^2 \qquad n = 1, 2, 3, \ldots \\ Y_n &= \sin(ny) \end{aligned}$$

$\boxed{\text{in } x:}$

$$\begin{aligned} X'' - \mu_n^2 X &= 0 \\ X(0) &= 0 \\ X(\pi) &= \text{undetermined} \end{aligned}$$

$$\Rightarrow X(x) = A\cosh(nx) + B\sinh(nx)$$

$$X(0) = 0 \Rightarrow A = 0 \Longrightarrow X = \underline{B\sinh(nx)}$$

$$\Rightarrow \underline{u(x,y) = \sum_{n=1}^{\infty} B_n \sinh(nx)\sin(ny)}$$

Boundary condition:

$$u(\pi, y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi)\sin(ny) = 100$$

$$\Rightarrow \underline{B_n \sinh(n\pi)} = \frac{2}{\pi}\int_0^{\pi} 100\sin(ny)\,dy = \frac{200}{\pi}\left(-\frac{\cos(ny)}{n}\right)\Big|_0^{\pi}$$

$$= \frac{200}{\pi n}\left(1 - \cos(n\pi)\right)$$

$$= \frac{200}{\pi n}\left(1 - (-1)^n\right) = \frac{200}{\pi n} \cdot \begin{cases} 0 & n = 2, 4, 6, \ldots \\ 2 & n = 1, 3, 5, 7, \ldots \end{cases}$$

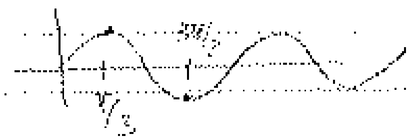
$$\Rightarrow \boxed{u(x,y) = \frac{400}{\pi} \sum_{k=0}^{\infty} \frac{1}{\sinh((2k+1)\pi)} \frac{\sinh((2k+1)x)\sin((2k+1)y)}{(2k+1)}}$$

b) $x = \frac{\pi}{2}, y = \frac{\pi}{2}$

$$\Rightarrow \underline{u\left(\frac{\pi}{2}, \frac{\pi}{2}\right)} = \frac{400}{\pi} \sum_{k=0}^{\infty} \frac{\sinh((2k+1)\frac{\pi}{2})}{\sinh((2k+1)\pi)} \frac{\overbrace{\sin((2k+1)\frac{\pi}{2})}^{(-1)^k}}{(2k+1)}$$

this

$$= \frac{1}{2\cosh((2k+1)\frac{\pi}{2})}$$



$$= \underline{\underline{\frac{200}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\cosh((2k+1)\frac{\pi}{2})}}}$$

Converges by the alternating sign series

Problem 6:

a) Inserting the solution of the form $u(x,t) = T(t)\sin(3\pi x)$ into the equation yields

$$\left(T'' + 2\pi T'\right) \cancel{\sin(3\pi x)} = -9\pi^2 \cancel{\sin(3\pi x)}$$

$$\Rightarrow T'' + 2\pi T' + 9\pi^2 T = 0$$

The characteristic equation is $r^2 + 2\pi r + 9\pi^2 = 0$. Thus,

$$r_{1,2} = \frac{-2\pi \pm \sqrt{4\pi^2 - 4\cdot 9\pi^2}}{2} = -\pi \pm i\pi\sqrt{8}$$

The general solution is

$$\underline{T(t) = e^{-\pi t}\left(A\sin(\sqrt{8}\pi t) + B\cos(\sqrt{8}\pi t)\right)}$$

b) To find $u(x,t)$, we also need $T(0) = 1$ and $T'(0) = 0$

$$T(0) = B \stackrel{!}{=} 1 \Rightarrow \underline{B = 1}$$

$$T'(t) = -\pi e^{-\pi t}\left(A\sin(\sqrt{8}\pi t) + B\cos(\sqrt{8}\pi t)\right)$$
$$+ e^{-\pi t}\left(\sqrt{8}A\pi \cos(\sqrt{8}\pi t) - B\sqrt{8}\pi \sin(\sqrt{8}\pi t)\right)$$

$$T'(0) = -\pi + \sqrt{8}A\pi \stackrel{!}{=} 0 \Rightarrow \underline{A = 1/\sqrt{8}}$$

$$\Rightarrow \underline{u(x,t) = e^{-\pi t}\left(\frac{\sin(\sqrt{8}\pi t)}{\sqrt{8}} + \cos(\sqrt{8}\pi t)\right)\sin(3\pi x)}$$